Partial solutions to Stochastic Calculus and Financial Applications by J. Michael Steele

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Problem (Exercise 6.1).

Proof. (i) We have

$$\mathrm{Var}\left[\int_0^t |B_s|^{1/2} dB_s\right] = \mathbb{E}\left[\left(\int_0^t |B_s|^{1/2} dB_s\right)^2\right] - \left(\mathbb{E}\left[\int_0^t |B_s|^{1/2} dB_s\right]\right)^2 \tag{1}$$

Since $\int_0^t |B_s|^{1/2} dB_s$ is a martingale, by property of martingale, we know it must has zero expectation, that is

$$\mathrm{Var}\left[\int_0^t |B_s|^{1/2} dB_s\right] = \mathbb{E}\left[\left(\int_0^t |B_s|^{1/2} dB_s\right)^2\right] \tag{2}$$

By Ito's Isometry, we have

$$\begin{aligned}
\mathbb{E}\left[\left(\int_0^t |B_s|^{1/2} dB_s\right)^2\right] &= \mathbb{E}\left[\int_0^t |B_s| ds\right] && (3)\\
&= \int_0^t \mathbb{E}|B_s| ds && (4)\\
&= \int_0^t \left(\int_{-\infty}^{\infty} |x| \cdot \frac{1}{\sqrt{2\pi s}} \cdot \exp(-x^2/2s) dx\right) ds && (5)\\
&= 2\int_0^t \left(\int_0^{\infty} x \cdot \frac{1}{\sqrt{2\pi s}} \cdot \exp(-x^2/2s) dx\right) ds && (6)\\
&= \frac{2}{\sqrt{2\pi}} \int_0^t \sqrt{s} ds && (7)\\
&= \frac{2}{\sqrt{2\pi}} \cdot \frac{2}{3} \cdot t^{3/2} = \frac{2\sqrt{2}}{3\sqrt{\pi}} t^{3/2} && (8)
\end{aligned}$$

so that

$$\mathrm{Var}\left[\int_0^t |B_s|^{1/2} dB_s\right] = \frac{2\sqrt{2}}{3\sqrt{\pi}} t^{3/2} \tag{9}$$

(ii) Similarly, we have

$$\mathrm{Var}\left[\int_0^t (B_s + s)^2 dB_s\right] = \mathbb{E}\left[\left(\int_0^t (B_s + s)^2 dB_s\right)^2\right] \tag{10}$$

then Ito's Isometry implies that

$$\begin{aligned}
\mathbb{E}\left[\left(\int_0^t (B_s+s)^2 dB_s\right)^2\right] &= \mathbb{E}\left[\int_0^t (B_s+s)^4 ds\right] && (11)\\
&= \int_0^t \mathbb{E}\left[(B_s+s)^4\right] ds && (12)\\
&= \int_0^t \mathbb{E}\left[4s^3B_s + 6s^2B_s^2 + 4sB_s^3 + B_s^4 + s^4\right] ds && (13)\\
&= \int_0^t 6s^3 + 3s^2 + s^4 ds && (14)\\
&= \frac{t^5}{5} + \frac{3t^4}{2} + t^3 && (15)
\end{aligned}$$

so that

$$\text{Var}\left[\int_0^t (B_s+s)^2 dB_s\right] = \frac{t^5}{5} + \frac{3t^4}{2} + t^3 \qquad (16)$$

as desired.

□

Problem (Exercise 8.2).

Proof. Define the function $f \in C^{1,2}(\mathbb{R}^+, \mathbb{R})$

$$f(t,x) := xh(t) \qquad (17)$$

Note that

$$\frac{\partial f}{\partial x} = h(t) \quad \frac{\partial f}{\partial t} = xh'(t) \quad \frac{\partial^2 f}{\partial x^2} = 0 \qquad (18)$$

By Ito's formula, we get

$$\begin{aligned}
h(t)B_t = f(t,B_t) &= f(0,B_0) + \int_0^t \frac{\partial f}{\partial x}(s,B_s)dB_s + \int_0^t \frac{\partial f}{\partial t}(s,B_s)dB_s + \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x^2}(s,B_s)ds && (19)\\
&= 0 + \int_0^t h(s)dB_s + \int_0^t h'(s)B_s dB_s + 0 && (20)\\
&= \int_0^t h(s)dB_s + \int_0^t h'(s)B_s dB_s && (21)
\end{aligned}$$

which proves the claim. □

Problem (Exercise 8.4).

Proof. (a) Assume the form $f(t,x) = \phi(t)\psi(x)$. First, we separate the variables, consider

$$0 = f_t + \frac{1}{2}f_{xx} = \phi'(t)\psi(x) + \frac{1}{2}\phi(t)\psi''(x) = 2\phi'(t)\psi(x) + \phi(t)\psi''(x) \qquad (22)$$

Rearrange the term and denote the separation constant as $-K$, we get

$$\frac{-2\phi'(t)}{\phi(t)} = \frac{\psi''(x)}{\psi(x)} = -K \qquad (23)$$

so that

$$\phi'(t) = \frac{K}{2}\cdot\phi(t) \quad \psi''(x) + K\cdot\psi(x) = 0 \qquad (24)$$

There are three cases depending on the value of K, consider:

(i) If $K = 0$, we get

$$\psi''(x) = 0 \implies \psi(x) = a + bx \tag{25}$$
$$\phi'(t) = 0 \implies \phi(t) = c \tag{26}$$

for some constant $a, b, c \in \mathbb{R}$ and

$$M_t = c(a + bB_t) \tag{27}$$

(ii) If $K > 0$, then we get

$$\psi(x) = a\cos\left(\sqrt{\lambda}x\right) + b\sin\left(\sqrt{\lambda}x\right) \quad \phi(t) = c\exp\left(\frac{k}{2}\lambda t\right) \tag{28}$$

for some constant a, b, c, then

$$M_t = c\left(a\cos\left(\sqrt{\lambda}B_t\right) + b\sin\left(\sqrt{\lambda}B_t\right)\right)\exp\left(\frac{k}{2}\lambda t\right) \tag{29}$$

(iii) If $K < 0$, then we get

$$\psi(x) = a\cosh\left(\sqrt{-\lambda}x\right) + b\sinh\left(\sqrt{-\lambda}x\right) \quad \phi(t) = c\exp\left(\frac{k}{2}\lambda t\right) \tag{30}$$

for some constant a, b, c, so that

$$M_t = c\left(a\cosh\left(\sqrt{-\lambda}B_t\right) + b\sinh\left(\sqrt{-\lambda}B_t\right)\right)\exp\left(\frac{k}{2}\lambda t\right) \tag{31}$$

which completes the proof.

(b) Apply Taylor's theorem up to 3rd order at zero, we get

$$M_t = 1 + B_t \cdot \alpha + \frac{1}{2}\left(B_t^2 - t\right) \cdot \alpha^2 + \frac{1}{6}\left(B_t^3 - 3tB_t\right) \cdot \alpha^3 + \cdots \tag{32}$$

It follows that the first four of $H_k(t, x)$ are

$$H_0(t,x) = 1 \quad H_1(t,x) = x \quad H_2(t,x) = \frac{1}{2}\left(x^2 - t\right) \quad H_3(t,x) = \frac{1}{6}\left(x^3 - 3tx\right) \tag{33}$$

Lastly, fix $k \in \mathbb{Z}^+$, we show $M_t(k) = H_k(t, B_t)$ is a martingale. We exploit the fact that M_t is a martingale. For $s < t$, we have

$$M_s = \mathbb{E}\left[M_t \mid F_s\right] \implies \sum_{k=0}^{\infty} \alpha^k H_k(s, B_s) = \mathbb{E}\left[\sum_{k=0}^{\infty} \alpha^k H_k(t, B_t) \mid F_s\right] \tag{34}$$
$$= \sum_{k=0}^{\infty} \alpha^k \mathbb{E}\left[H_k(t, B_t) \mid F_s\right] \tag{35}$$

It follows that

$$\mathbb{E}\left[H_k(t, B_t) \mid F_s\right] = H_k(s, B_s) \tag{36}$$

which proves the claim, as desired.

□

Problem (Exercise 8.5).

Proof. (a) First, we show f is Laplacian for $(x, y, z) \neq 0$. Consider

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \tag{37}$$

in which

$$\frac{\partial^2 f}{\partial x^2} = \frac{2x^2 - y^2 - z^2}{(x^2+y^2+z^2)^{5/2}} \quad \frac{\partial^2 f}{\partial y^2} = \frac{-x^2 + 2y^2 - z^2}{(x^2+y^2+z^2)^{5/2}} \quad \frac{\partial^2 f}{\partial z^2} = \frac{-x^2 - y^2 + 2z^2}{(x^2+y^2+z^2)^{5/2}} \tag{38}$$

Note that the above numerators sum up to zero, so that we have $\Delta f = 0$ and f is harmonic. By Proposition 8.3, we immediately have M_t to be a local martingale.

(b) Denote

$$\vec{B}_t = \left(B_t^1, B_t^2, B_t^3\right) \in \mathbb{R}^3 \tag{39}$$

where $B_t^i \sim \mathcal{N}(0, t)$ for each $i \in \{1, 2, 3\}$. Observe

$$\mathbb{E}(M_t^2) = \mathbb{E}\left(f(\vec{B}_t)^2\right) = \mathbb{E}\left(\left(\frac{1}{\sqrt{{B_t^1}^2 + {B_t^2}^2 + {B_t^3}^2}}\right)^2\right) \tag{40}$$

$$= \mathbb{E}\left(\frac{1}{{B_t^1}^2 + {B_t^2}^2 + {B_t^3}^2}\right) \tag{41}$$

then with spherical coordinates, we get

$$\mathbb{E}\left(\frac{1}{{B_t^1}^2 + {B_t^2}^2 + {B_t^3}^2}\right) = \frac{1}{\left(\sqrt{2\pi t}\right)^3} \iiint_{\mathbb{R}^3} \frac{1}{x^2+y^2+z^2} \cdot \exp\left(-\frac{x^2+y^2+z^2}{2t}\right) dV \tag{42}$$

$$= \frac{1}{\left(\sqrt{2\pi t}\right)^3} \int_0^{\pi} \int_0^{2\pi} \int_0^{\infty} \rho^2 \sin\varphi \cdot \frac{1}{\rho^2} \cdot \exp\left(-\frac{\rho^2}{2t}\right) d\rho d\theta d\varphi \tag{43}$$

$$= \frac{1}{\left(\sqrt{2\pi t}\right)^3} \int_0^{\pi} \int_0^{2\pi} \int_0^{\infty} \sin\varphi \cdot \exp\left(-\frac{\rho^2}{2t}\right) d\rho d\theta d\varphi \tag{44}$$

$$= \frac{2\pi}{\left(\sqrt{2\pi t}\right)^3} \int_0^{\pi} \int_0^{\infty} \sin\varphi \cdot \exp\left(-\frac{\rho^2}{2t}\right) d\rho d\varphi \tag{45}$$

$$= \frac{4\pi}{\left(\sqrt{2\pi t}\right)^3} \int_0^{\infty} \exp\left(-\frac{\rho^2}{2t}\right) d\rho \tag{46}$$

$$= \frac{4\pi}{\left(\sqrt{2\pi t}\right)^3} \cdot \sqrt{\frac{\pi t}{2}} = \frac{4\pi}{\sqrt{2\pi t}} \cdot \frac{\sqrt{2\pi t}}{2} \cdot \frac{1}{2\pi t} = \frac{1}{t} \tag{47}$$

which proves the claim.

(c) For the sake of contradiction, suppose that M_t is a martingale, then given $s < t$, we have

$$\mathbb{E}\left[M_t \mid \mathcal{F}_s\right] = M_s \tag{48}$$

Consider the convex function $\varphi(x) = x^2$, then Jensen's inequality implies that

$$\mathbb{E}\left[M_t^2 \mid F_s\right] \geq \left(\mathbb{E}\left[M_t \mid \mathcal{F}_s\right]\right)^2 = M_s^2 \tag{49}$$

Take expectation of both sides, part (b) implies that

$$\mathbb{E}M_t^2 \geq \mathbb{E}M_s^2 \implies \frac{1}{t} \geq \frac{1}{s} \tag{50}$$

However, since we set $s < t$, the above is absurd, contradiction.

□

Problem (Exercise 8.3).

Proof. (a) By Cauchy-Riemann equation, we have

$$u_x - v_y = 0 \implies u_{xx} - v_{yx} = 0 \tag{51}$$
$$u_y + v_x = 0 \implies u_{yy} + v_{xy} = 0 \tag{52}$$

Sum up the two equations, we get

$$\Delta u = u_{xx} + u_{yy} = 0 \tag{53}$$

so that u is harmonic. Similarly, we have

$$u_x - v_y = 0 \implies u_{xy} - v_{yy} = 0 \tag{54}$$
$$u_y + v_x = 0 \implies u_{yx} + v_{xx} = 0 \tag{55}$$

so that minus the two yields

$$\Delta v = v_{xx} + v_{yy} = 0 \tag{56}$$

and v is harmonic. Now, we decompose the two analytic functions by Euler's formula

(i) Since $\exp(z) = e^{x+iy} = e^x (\cos y + i \sin y)$, then we obtain

$$\mathrm{Re}(\exp(z)) = e^x \cos y \quad \mathrm{Im}(\exp(z)) = e^x \sin y \tag{57}$$

(ii) Note that

$$z \exp(z) = (x + iy)e^x (\cos y + i \sin y) \tag{58}$$
$$= xe^x \cos y + ixe^x \sin y + iye^x \cos y - ye^x \sin y \tag{59}$$

then we get

$$\mathrm{Re}(z \exp(z)) = e^x (x \cos y - y \sin y) \quad \mathrm{Im}(z \exp(z)) = e^x (x \sin y + y \cos y) \tag{60}$$

as desired.

(b) First, we decompose the analytic function $f(z) = z^2$. Consider

$$f(z) = (x + iy)^2 = (x^2 - y^2) + i (2xy) \tag{61}$$

then it follows that $\mathrm{Re}(f) = u(x, y) = x^2 - y^2$ is harmonic from part (a). Consequently, we have $X_t := u(\vec{B}_t)$ as a local martingale with $X_0 = 4$. Define the stopping times

$$\tau_1 = \inf\{t : X_t = 1\} \quad \tau_5 = \inf\{t : X_t = 5\} \tag{62}$$

and $\tau = \tau_1 \wedge \tau_5$. Alternatively, we may express τ as

$$\tau = \inf\{t : X_t - X_0 = 1 \text{ or } X_t - X_0 = -3\} \tag{63}$$

Now, we apply Proposition 7.8 on $X_t - X_0$ to compute $\mathbb{P}_{(2,0)} (X_\tau = 1)$, but we need to justify why we may apply it. Since $\vec{B}_t \in \mathbb{R}^2$ is recurrent, then we know $\tau < \infty$ almost surely. Then, Proposition 7.8 is justified and

$$\mathbb{P}_{(2,0)} (X_\tau = 1) = \mathbb{P}_{(2,0)} (X_\tau - X_0 = -3) = 1 - \frac{3}{1 + 3} = \frac{1}{4} \tag{64}$$

as desired.

□