Peter-Weyl Theorem

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Math 520: Representation Theory

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Outline

The problem

The problem that motivates Peter-Weyl Theorem is the representation of "topological groups" in the space of functions. Namely, we are considering some "locally compact group" G and the vector space $L^2(G)$.

Locally Compact Group

Definition

A group G with identity 1 is a topological group if G is a Hausdorff topological space and the map

$$G \times G \to G : (g, h) \mapsto gh^{-1}$$

is continuous.

Definition

A topological space X is locally compact if and only if for every point $p \in X$, there is a compact neighborhood C of p; that is, there is a compact C and an open U, with $p \in U \subset C$. For example, the additive group $(\mathbb{R}, +)$ is locally compact.

Haar Measure

Theorem

Every locally compact group G admits a positive Borel measure μ, called the Haar measure, where

$$\mu(g \cdot X) = \mu(X) = \mu(X \cdot g)$$

for all $g \in G$ and μ-measurable sets X.

L^p-Space

Definition

Suppose X is any arbitrary measure space with measure μ. If $0 < p < \infty$ and if f is a complex measurable function on X, define

$$\|f\|_p = \left(\int_X |f|^p d\mu\right)^{1/p}$$

and let $L^p(X)$ consist of all f for which

$$\|f\|_p < \infty$$

We call $\|f\|_p$ the L^p-norm of f.

At this point, we may introduce the Hilbert Space $L^2(G)$, space of square-integrable functions defined on G, and its inner product is given by

$$\langle f_1, f_2\rangle_{L^2} = \int_G f_1(g)\overline{f_2(g)}d\mu(g)$$

When G is finite

A special case is when our group G is a finite group.
Recall the regular representation $\mathbb{C}[G]$, where the vector space consists of formal expressions of the form

$$\sum_{g \in G} \lambda_g g$$

and acting by $h \in G$ just amounts to a left translation:

$$\rho(h)\left(\sum_{g \in G} \lambda_g g\right) = \sum_{g \in G} \lambda_g hg$$

When G is finite, we have $L^2(G) \cong \mathbb{C}[G]$ via the ismorphism $f \mapsto \sum_{g \in G} f(g)g$.
When G is infinite, $\mathbb{C}[G]$ becomes way too complicated to study.

How does G act on $L^2(G)$?

We can define action by G on $L^2(G)$ by

$$g \cdot f(h) = f(g^{-1}h)$$

or (left vs right)

$$g \cdot f(h) = f(hg)$$

(when G is compact, it turns out that there is no difference between the two).

Representation Theory

Theorem (Orthogonality Relations)

Let V_ρ and V_π be two nonisomorphic irreducible representations of G with respective G-invariant inner products $\langle -, - \rangle_\rho, \langle -, - \rangle_\pi$. Then for all $a, v \in V_\rho, b, w \in V_\pi$

$$\int_G \overline{\langle ga, v \rangle_\rho} \langle gb, w \rangle_\pi dg = 0$$

Note that in the analysis case, the character of a representation is replaced by this "inner product object" (matrix coefficient).

Theorem (Mashke)

Let V be a representation of the compact group G. If U is a subrepresentation of V, then there exists a subrepresentation W of V such that $V = U \oplus W$.

Matrix Coefficient: Definition

Definition

Given a finite dimensional representation $\rho : G \to GL(V_\rho)$, and some G-invariant inner product on V_ρ, we can define the corresponding **matrix coefficient**, which is just a function $\sigma_\rho : G \to \mathbb{C}$ such that $\sigma_\rho(g) = \langle \rho(g) v_1, v_2 \rangle_\rho$ for some $v_1, v_2 \in V_\rho$. We denote the set of matrix coefficient of G as $\mathcal{M}$.

Matrix Coefficient

Definition

For a particular irreducible representation $\rho : G \to GL(V_\rho)$, we can define the subspace

$$\mathcal{M}_\rho = Span\{\sigma_{\rho,v_1,v_2} \in \mathcal{M} : v_1, v_2 \in V_\rho\}$$

which is just the span of all of the matrix coefficients associated to the irreducible representation ρ. An element of $\mathcal{M}_\rho$ is called a **matrix coefficient of the representation** ρ.

Matrix Coefficient: Property

Proposition

Denote $\mathcal{M}$ as the set of matrix coefficients of G. $\mathcal{M}$ is closed under pointwise addition and scalar multiplication.

Proposition

Let $[\rho]$ denote the an equivalence class of isomorphic representations of G. If $\pi \in [\rho]$, then we know $\mathcal{M}_\pi = \mathcal{M}_\rho$.

Matrix Coefficient: $\bigoplus$

From Mashke, we know a matrix coefficient of a reducible representation is a sum of matrix coefficients of irreducible representations.
By Schur's Orthogonality relation, we have

$$\mathcal{M} = \bigoplus_{[\rho]} \mathcal{M}_{[\rho]}$$

with summands orthogonal with respect to the $L^2(G)$ inner product.

Matrix Coefficient: $\otimes$

Proposition

Let $V_\rho \otimes V_\rho^*$ be the representation of G under the aciton

$$g(v_1 \otimes \langle -, v_2 \rangle_\rho) = gv_1 \otimes \langle -, v_2 \rangle_\rho$$

then

$$\mathcal{M} \cong \bigoplus_{[\rho]} \mathcal{M}_{[\rho]} \cong \bigoplus_{[\rho]} V_\rho \otimes V_\rho^* \cong \bigoplus_{[\rho]} V_\rho^{\dim V_\rho}$$

A black box that ties everything together

Theorem

The set of matrix coefficients of G is dense in $C(G, \mathbb{C})$ (the subspace of continuous functions)

The Peter-Weyl Theorem

Theorem

(Peter-Weyl Theorem Part I). As a representation of G,

$$L^2(G) \cong \overline{\mathcal{M}} \cong \overline{\bigoplus_{[\rho]} V_\rho^{dim V_\rho}} = \widehat{\bigoplus_{[\rho]}} V_\rho^{dim V_\rho}$$

To be clear, we are summing over all irreducible representation classes of G.

Example of Peter-Weyl Theorem in action

Consider $G = S^1 = \{z = r \cdot e^{ix} \in \mathbb{C} : |z| = 1\}$ Abelian group.
Therefore, Schur's Lemma indicates that ρ maps some element of S^1 into $\mathbb{C}^\times$.
Know $\rho(z) \in S^1$ through some non-trivial results.
Implies all representations of S^1 are continuous homomorphisms $S^1 \to S^1$, so must be of the form $e^{ix} \longmapsto e^{ixn}$ where $n \in \mathbb{Z}$.
Therefore, $\rho_n(e^{ix}) \in GL(\mathbb{C})$ maps $z \in \mathbb{C}$ to ze^{ixn}.
Matrix coefficients are of form

$$\sigma_n(e^{ix}) = \langle \rho_n(e^{ix}) z_1, z_2 \rangle = z_1 \bar{z}_2 e^{ixn}$$

So, $\mathcal{M}_{[\rho_n]} = \mathbb{C}e^{ixn}$, and Peter-Weyl gives

$$L^2(S^1) = \widehat{\bigoplus_{[\rho_n]}} \mathbb{C}e^{nix}$$

References

1. Jean Gallier and Jocelyn Quaintance, Aspects of Harmonic Analysis and Representation Theory
2. Walter Rudin, Real and Complex Analysis
3. John Bergan, The Peter-Weyl Theorem and Generalization
4. Terence Tao, The Peter-Weyl theorem, and non-abelian Fourier analysis on compact groups