Partial solutions to Probability: Theory and Examples by Rick Durrett

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Problem 4.1.7.

Proof. First, observe that

$$\begin{aligned} \mathrm{Var} X &= \mathbb{E}X^2 - (\mathbb{E}X)^2 && (1) \\ &= \mathbb{E}\left[\mathbb{E}\left[X^2 \mid \mathcal{F}\right]\right] - (\mathbb{E}\left[\mathbb{E}\left[X \mid \mathcal{F}\right]\right])^2 && (2) \end{aligned}$$

Note that we may further manipulate the above term using assumption, that is

$$\begin{aligned} \mathbb{E}\left[\mathbb{E}\left[X^2 \mid \mathcal{F}\right]\right] &= \mathbb{E}\left[\mathrm{Var}\left[X \mid \mathcal{F}\right] + (\mathbb{E}\left[X \mid \mathcal{F}\right])^2\right] && (3) \\ &= \mathbb{E}\left[\mathrm{Var}\left[X \mid \mathcal{F}\right]\right] + \mathbb{E}\left[(\mathbb{E}\left[X \mid \mathcal{F}\right])^2\right] && (4) \end{aligned}$$

then we have

$$\begin{aligned} \mathrm{Var} X &= \mathbb{E}\left[\mathrm{Var}\left[X \mid \mathcal{F}\right]\right] + \mathbb{E}\left[(\mathbb{E}\left[X \mid \mathcal{F}\right])^2\right] - (\mathbb{E}\left[\mathbb{E}\left[X \mid \mathcal{F}\right]\right])^2 && (5) \\ &= \mathbb{E}\left[\mathrm{Var}\left[X \mid \mathcal{F}\right]\right] + \left(\mathbb{E}\left[(\mathbb{E}\left[X \mid \mathcal{F}\right])^2\right] - (\mathbb{E}\left[\mathbb{E}\left[X \mid \mathcal{F}\right]\right])^2\right) && (6) \\ &= \mathbb{E}\left[\mathrm{Var}\left[X \mid \mathcal{F}\right]\right] + \mathrm{Var}\left[\mathbb{E}\left[X \mid \mathcal{F}\right]\right] && (7) \end{aligned}$$

as desired. □

Problem 4.1.8.

Proof. For the set up, we define $\mathcal{F} = \sigma(N)$, then consider the following identity from the definition of conditional expectation

$$\begin{aligned} \mathrm{Var} X &= \mathbb{E}X^2 - (\mathbb{E}X)^2 && (8) \\ &= \mathbb{E}\left[\mathbb{E}\left[X^2 \mid \mathcal{F}\right]\right] - (\mathbb{E}\left[\mathbb{E}\left[X \mid \mathcal{F}\right]\right])^2 && (9) \end{aligned}$$

Now, we compute $\mathbb{E}\left[X \mid \mathcal{F}\right]$ and $\mathbb{E}\left[X^2 \mid \mathcal{F}\right]$. Denote the event $N = n$ as Ω_n, where $n \in \mathbb{Z}^+$, then $\bigcup_{n\in\mathbb{Z}^+} \Omega_n = \Omega$, and Ω_n's are disjoint. Observe that

$$\begin{aligned} \int_{\Omega_n} X d\mathbb{P} &= \int_{\Omega_n} (Y_1 + \cdots + Y_n)\, d\mathbb{P} && (10) \\ &= n\mathbb{E}\left[Y_1\right]\mathbb{E}\left[\mathbf{1}_{\Omega_n}\right] && (11) \\ &= n\mu\mathbb{P}\left[N = n\right] && (12) \\ &= \int_{\Omega_n} N\mu d\mathbb{P} && (13) \end{aligned}$$

holds for all $\Omega_n \in \mathcal{F}$, then we have $\mathbb{E}[X \mid \mathcal{F}] = N\mu$. Similarly, consider

$$\begin{aligned}
\int_{\Omega_n} X^2 d\mathbb{P} &= \int_{\Omega_n} (Y_1 + \cdots + Y_n)^2 \, d\mathbb{P} && (14)\\
&= \mathbb{E}\left[(Y_1 + \cdots + Y_n)^2 \cdot \mathbf{1}_{\Omega_n}\right] && (15)\\
&= \mathbb{E}\left[(Y_1 + \cdots + Y_n)^2\right] \cdot \mathbb{E}\left[\mathbf{1}_{\Omega_n}\right] && (16)\\
&= \left(\mathrm{Var}\left[Y_1 + \cdots + Y_n\right] + \mathbb{E}(Y_1 + \cdots + Y_n)^2\right) \cdot \mathbb{E}\left[\mathbf{1}_{\Omega_n}\right] && (17)\\
&= \left(n\sigma^2 + n^2\mu^2\right) \cdot \mathbb{P}\left[N = n\right] && (18)\\
&= \int_{\Omega_n} \left(N\sigma^2 + N^2\mu^2\right) d\mathbb{P} && (19)
\end{aligned}$$

so that $\mathbb{E}\left[X^2 \mid \mathcal{F}\right] = N\sigma^2 + N^2\mu^2$. Lastly, using equation 9, we get

$$\begin{aligned}
\mathrm{Var}X &= \mathbb{E}\left[\mathbb{E}\left[X^2 \mid \mathcal{F}\right]\right] - \left(\mathbb{E}\left[\mathbb{E}\left[X \mid \mathcal{F}\right]\right]\right)^2 && (20)\\
&= \mathbb{E}\left[N\sigma^2 + N^2\mu^2\right] - \left(\mathbb{E}\left[N\mu\right]\right)^2 && (21)\\
&= \sigma^2\mathbb{E}N + \mu^2\mathbb{E}N^2 - \mu^2(\mathbb{E}N)^2 && (22)\\
&= \sigma^2\mathbb{E}N + \mu^2 \cdot \left(\mathbb{E}N^2 - (\mathbb{E}N)^2\right) && (23)\\
&= \sigma^2\mathbb{E}N + \mu^2\mathrm{Var}N && (24)
\end{aligned}$$

as desired. □

Problem 4.2.9.

Proof. Since the other two conditions are satisfied by construction of Z_n and Y_n and the fact that X_n^1, X_n^2 are supermartingales., it suffices to show i) $\mathbb{E}\left[Y_{n+1} \mid \mathcal{F}_n\right] \le Y_n$ and ii) $\mathbb{E}\left[Z_{n+1} \mid \mathcal{F}_n\right] \le Z_n$. Consider:

i) Observe that

$$X_N^1 \ge X_N^2 \implies X_N^1 - X_N^2 \ge 0 \quad (25)$$

Consider

$$\begin{aligned}
Y_{n+1} &= X_{n+1}^1\mathbf{1}_{N>n+1} + X_{n+1}^2\mathbf{1}_{N\le n+1} && (26)\\
&= X_{n+1}^1\mathbf{1}_{N>n} - X_{n+1}^1\mathbf{1}_{N=n+1} + X_{n+1}^2\mathbf{1}_{N=n+1} + X_{n+1}^2\mathbf{1}_{N\le n} && (27)\\
&= X_{n+1}^1\mathbf{1}_{N>n} + X_{n+1}^2\mathbf{1}_{N\le n} - \left(X_{n+1}^1\mathbf{1}_{N=n+1} - X_{n+1}^2\mathbf{1}_{N=n+1}\right) && (28)\\
&\le X_{n+1}^1\mathbf{1}_{N>n} + X_{n+1}^2\mathbf{1}_{N\le n} && (29)
\end{aligned}$$

Take conditional expectation of both sides and use the fact that X_n^1, X_n^2 are supermartingales, we get

$$\begin{aligned}
\mathbb{E}\left[Y_{n+1} \mid \mathcal{F}_n\right] &\le \mathbb{E}\left[X_{n+1}^1\mathbf{1}_{N>n} + X_{n+1}^2\mathbf{1}_{N\le n} \mid \mathcal{F}_n\right] && (30)\\
&= \mathbb{E}\left[X_{n+1}^1 \mid \mathcal{F}_n\right]\mathbf{1}_{N>n} + \mathbb{E}\left[X_{n+1}^2 \mid \mathcal{F}_n\right]\mathbf{1}_{N\le n} && (31)\\
&\le X_n^1\mathbf{1}_{N>n} + X_n^2\mathbf{1}_{N\le n} = Y_n && (32)
\end{aligned}$$

ii) Similarly, we have

$$\begin{aligned}
Z_{n+1} &= X_{n+1}^1\mathbf{1}_{N\ge n+1} + X_{n+1}^2\mathbf{1}_{N<n+1} && (33)\\
&= \left(X_{n+1}^1\mathbf{1}_{N>n+1} + X_{n+1}^2\mathbf{1}_{N\le n+1}\right) + \left(X_{n+1}^1\mathbf{1}_{N=n+1} - X_{n+1}^2\mathbf{1}_{N=n+1}\right) && (34)\\
&= Y_{n+1} + \left(X_{n+1}^1\mathbf{1}_{N=n+1} - X_{n+1}^2\mathbf{1}_{N=n+1}\right) && (35)\\
&= X_{n+1}^1\mathbf{1}_{N>n} + X_{n+1}^2\mathbf{1}_{N\le n} \quad \text{by applying equation 28 on } Y_{n+1} && (36)
\end{aligned}$$

Take conditional expectation of both side, we get

$$\mathbb{E}\left[Z_{n+1} \mid \mathcal{F}_n\right] \leq X_n^1 \mathbf{1}_{N>n} + X_n^2 \mathbf{1}_{N\leq n} = Y_n \tag{37}$$

Lastly, observe that

$$Z_n - Y_n = X_n^1 \mathbf{1}_{N=n} - X_n^2 \mathbf{1}_{N=n} \geq 0 \tag{38}$$

It immediately follows from equation 37 that

$$\mathbb{E}\left[Z_{n+1} \mid \mathcal{F}_n\right] \leq Y_n \leq Z_n \tag{39}$$

as desired. □

Problem 4.2.10.

Proof. (i) We use switching principle to prove the first assertion. For the base step, when $j = 1$, we have

$$Z_n^1 = Y_{n\wedge N_1} = Y_n \mathbf{1}_{n<N_1} + Y_{N_1} \mathbf{1}_{n\geq N_1} \tag{40}$$

$$= 1 \cdot \mathbf{1}_{N_1>n} + \frac{X_n}{a} \cdot \mathbf{1}_{N_1\leq n} \tag{41}$$

Note that at N_1, from the assumption, for all $m > N_0 = -1$, we have

$$X_{N_1} \leq a \implies \frac{X_{N_1}}{a} \leq 1 \tag{42}$$

Thus, we may conclude from switching principle that Z_n^1 is a supermartingale.

Before the induction step, we shall make some observation for Z_n^2, Z_n^3. Consider

$$Z_n^2 = Y_{n\wedge N_2} = Y_n \mathbf{1}_{n<N_2} + Y_{N_2} \mathbf{1}_{n\geq N_2} \tag{43}$$

$$= \left(Y_n \mathbf{1}_{n<N_1} + Y_n \mathbf{1}_{n\geq N_1}\right) \mathbf{1}_{n<N_2} + \frac{b}{a} \cdot \mathbf{1}_{n\geq N_2} \tag{44}$$

$$= \left(1 \cdot \mathbf{1}_{n<N_1} + (X_n/a) \mathbf{1}_{n\geq N_1}\right) \mathbf{1}_{n<N_2} + (b/a) \mathbf{1}_{n\geq N_2} \tag{45}$$

$$= Z_n^1 \mathbf{1}_{n<N_2} + (b/a) \mathbf{1}_{n\geq N_2} \tag{46}$$

Note that at N_2, for all $m > N_1$, we have

$$X_{N_2} \geq b \implies \frac{X_{N_2}}{a} \geq \frac{b}{a} \implies Z_{N_2}^1 \geq (b/a) \tag{47}$$

so that we may apply switching principle to conclude that Z_n^2 is a supermartingale. For Z_n^3, we have

$$Z_n^3 = Y_{n\wedge N_3} = Y_n \mathbf{1}_{n<N_3} + Y_{N_3} \mathbf{1}_{n\geq N_3} \tag{48}$$

$$= \left(Y_n \mathbf{1}_{n<N_2} + Y_n \mathbf{1}_{n\geq N_2}\right) \mathbf{1}_{n<N_3} + (b/a)(X_{N_3}/a) \mathbf{1}_{n\geq N_3} \tag{49}$$

$$= \left[\left(1 \cdot \mathbf{1}_{n<N_1} + (X_n/a) \mathbf{1}_{n\geq N_1}\right) \mathbf{1}_{n<N_2} + (b/a) \mathbf{1}_{n\geq N_2}\right] \mathbf{1}_{n<N_3} + (b/a)(X_{N_3}/a) \mathbf{1}_{n\geq N_3} \tag{50}$$

$$= \left[Z_n^1 \mathbf{1}_{n<N_2} + (b/a) \mathbf{1}_{n\geq N_2}\right] \mathbf{1}_{n<N_3} + (b/a)(X_{N_3}/a) \mathbf{1}_{n\geq N_3} \tag{51}$$

$$= Z_n^2 \mathbf{1}_{n<N_3} + (b/a)(X_{N_3}/a) \mathbf{1}_{n\geq N_3} \tag{52}$$

Note that at N_3, for all $m > N_2$, we have

$$X_{N_3} \leq a \implies (b/a)(X_{N_3}/a) \leq (b/a) \implies Z_{N_3}^2 \geq (b/a)(X_{N_3}/a) \tag{53}$$

so that switching principle implies Z_n^3 is a supermartingale. Recursively apply the above, for $k \geq 1$, we have

$$Z_n^{2k} = Z_n^{2k-1} \mathbf{1}_{n<N_{2k}} + (b/a)^k \cdot \mathbf{1}_{n\geq N_{2k}} \tag{54}$$

$$Z_n^{2k+1} = Z_n^{2k} \mathbf{1}_{n<N_{2k+1}} + (b/a)^k (X_{N_{2k+1}}/a) \mathbf{1}_{n\geq N_{2k+1}} \tag{55}$$

For the induction step, assume $Z^{2k-1}_{N_{2k}} \geq (b/a)^k$ so that Z^{2k}_n is a supermartingale by switching principle. Consider

$$\begin{aligned} Z^{2k+1}_n &= Z^{2k}_n \mathbf{1}_{n<N_{2k+1}} + (b/a)^k (X_{N_{2k+1}}/a) \mathbf{1}_{n \geq N_{2k+1}} && (56) \\ &= \left(Z^{2k-1}_n \mathbf{1}_{n<N_{2k}} + (b/a)^k \cdot \mathbf{1}_{n \geq N_{2k}} \right) \mathbf{1}_{n<N_{2k+1}} + (b/a)^k (X_{N_{2k+1}}/a) \mathbf{1}_{n \geq N_{2k+1}} && (57) \end{aligned}$$

At N_{2k+1}, for all $m > N_{2k}$, combined with assumption from induction, we have

$$\begin{aligned} X_{N_{2k+1}} \leq a &\implies (b/a)^k \left(X_{N_{2k+1}}/a \right) \leq (b/a)^k && (58) \\ &\implies \left(Z^{2k-1}_{N_{2k+1}} \mathbf{1}_{n<N_{2k}} + (b/a)^k \cdot \mathbf{1}_{n \geq N_{2k}} \right) \geq (b/a)^k (X_{N_{2k+1}}/a) && (59) \end{aligned}$$

so that the switching principle implies Z^{2k+1}_n is a supermartingale. Apply the process recursively, we can deduce that Z^{2k+2}_n is also a supermartingale by the switching principle, as desired.

(ii) By part (i), we know that Z^j_n is supermartingale, so that

$$\mathbb{E} Z^{2k}_n = \mathbb{E} Y_{n \wedge N_{2k}} \leq \mathbb{E} Y_0 \tag{60}$$

Observe that when $0 \leq n < N_1$, we have $Y_0 = 1$. If $N_1 = 0$, then we must have $X_0 \leq a$. It follows that

$$Y_0 = \min(X_0/a, 1) \implies \mathbb{E} Y_0 = \mathbb{E} \min(X_0/a, 1) \tag{61}$$

Consider

$$\begin{aligned} \lim_{n\to\infty} \mathbb{E} Y_{n \wedge N_{2k}} &= \lim_{n\to\infty} \mathbb{E} Y_{n \wedge N_{2k}} && (62) \\ &= \lim_{n\to\infty} \mathbb{E} \left[Y_{n \wedge N_{2k}} \mathbf{1}_{U<k} + Y_{n \wedge N_{2k}} \mathbf{1}_{U \geq k} \right] && (63) \\ &= \lim_{n\to\infty} \mathbb{E} \left[\left(Y_n \mathbf{1}_{n<N_{2k}} + Y_{N_{2k}} \mathbf{1}_{n \geq N_{2k}} \right) \mathbf{1}_{U<k} \right] && (64) \\ &+ \lim_{n\to\infty} \mathbb{E} \left[\left(Y_n \mathbf{1}_{n<N_{2k}} + Y_{N_{2k}} \mathbf{1}_{n \geq N_{2k}} \right) \mathbf{1}_{U \geq k} \right] && (65) \end{aligned}$$

Note that Y_n is positive, then from equation 64, we get

$$\begin{aligned} \lim_{n\to\infty} \mathbb{E} \left[\left(Y_n \mathbf{1}_{n<N_{2k}} + Y_{N_{2k}} \mathbf{1}_{n \geq N_{2k}} \right) \mathbf{1}_{U<k} \right] &= \mathbb{E} \left[\lim_{n\to\infty} Y_n \mathbf{1}_{U<k} \right] && (66) \\ &= \mathbb{E} \left[\lim_{n\to\infty} Y_n \right] \mathbb{P} \left[U < k \right] \geq 0 && (67) \end{aligned}$$

and from equation 65

$$\begin{aligned} \lim_{n\to\infty} \mathbb{E} \left[\left(Y_n \mathbf{1}_{n<N_{2k}} + Y_{N_{2k}} \mathbf{1}_{n \geq N_{2k}} \right) \mathbf{1}_{U \geq k} \right] &= \mathbb{E} \left[Y_{N_{2k}} \mathbf{1}_{U \geq k} \right] && (68) \\ &= (b/a)^j \mathbb{P} \left[U \geq k \right] && (69) \end{aligned}$$

Assemble the above result, we get

$$\lim_{n\to\infty} \mathbb{E} Y_{n \wedge N_{2k}} = (b/a)^j \mathbb{P} \left[U \geq k \right] + \mathbb{E} \left[\lim_{n\to\infty} Y_n \right] \mathbb{P} \left[U < k \right] \leq \mathbb{E} \min(X_0/a, 1) \tag{70}$$

which implies that

$$(b/a)^j \mathbb{P} \left[U \geq k \right] \leq \mathbb{E} \min(X_0/a, 1) \tag{71}$$

as desired.

□

Problem 4.2.4.

Proof. For the set up, define the stopping time

$$N_m = \inf\{k : X_k > m\} \tag{72}$$

for each $m \geq 0$. By Theorem 4.2.9, we know that $X_{N_m \wedge n}$ is a sub-martingale.

Now, we wish to show $X_{N_m \wedge n}$ converges a.s. using Theorem 4.2.11. Note that

$$\begin{aligned} \sup X^+_{N_m \wedge n} \leq X^+_{N_m} &= (\xi_{N_m} + X_{N_m - 1})^+ & (73)\\ &\leq \xi^+_{N_m} + X^+_{N_m - 1} & (74)\\ &\leq \sup \xi^+_{N_m} + m & (75) \end{aligned}$$

It follows that

$$\sup \mathbb{E} X^+_{N_m \wedge n} \leq \mathbb{E} \sup X^+_{N_m \wedge n} \leq \mathbb{E}\left(\sup \xi^+_{N_m}\right) + m < \infty \tag{76}$$

then we immediately have $X_{N_m \wedge n}$ converges a.s.. In particular, $X_{N_m \wedge n}$ converges on the event $\omega_m = \{N_m = \infty\}$, i.e., $X_{N_m \wedge n} \mathbf{1}_{N_m = \infty} = X_n$ converges. Since we have $\sup X_n < \infty$, then it must be the case that

$$\bigcup_{m=1}^{\infty} \omega_m = \Omega \tag{77}$$

Thus, X_n converges a.s., as desired.

□

Problem 4.2.6 (i).

Proof. Note that X_n is a non-negative martingale, in particular, by construction it is a super-martingale. Thus, we may apply Theorem 4.2.12 and conclude that there exists some random variable X with

$$X_n \to X \text{ a.s. and } \mathbb{E}X \leq \mathbb{E}X_0 \tag{78}$$

Choose $\delta > 0$ such that

$$\mathbb{P}\left(|Y_1 - 1| \geq \delta\right) > 0 \tag{79}$$

then for all $\epsilon > 0$, we have

$$\begin{aligned} \mathbb{P}\left(|X_{n+1} - X_n| \geq \epsilon\delta\right) &= \mathbb{P}\left(X_n |Y_{n+1} - 1| \geq \epsilon\delta\right) & (80)\\ &\geq \mathbb{P}\left(X_n \geq \epsilon\right) \mathbb{P}\left(|Y_{n+1} - 1| \geq \delta\right) & (81) \end{aligned}$$

Take limit of both sides, then note LHS tends to zero as $n \to \infty$. Since equation 79 is positive, then it must be the case that

$$\mathbb{P}\left(X_n \geq \epsilon\right) \to 0 \tag{82}$$

for all $\epsilon > 0$, which implies that $X = 0$ a.s., as desired. □

Problem 4.3.11.

Proof. If $\mu \leq 1$, then we have

$$\lim Z_n/\mu^n = 0 \implies \mathbb{P}\left(\lim Z_n/\mu^n = 0\right) = 1 \tag{83}$$

Thus, it must be the case that $\mu > 1$.

Since $\mu > 1$ and $Z_0 = 1$, then by Theorem 4.3.12, we have

$$\rho := \mathbb{P}\left(Z_n = 0 \text{ for some n}\right) \tag{84}$$

to be the only solution of $\varphi(\rho) = \rho \in [0, 1)$. Denote

$$\delta := \mathbb{P}\left(\lim Z_n/\mu^n = 0\right) \tag{85}$$
$$\delta_m := \mathbb{P}\left(Z_m/\mu^m = 0\right) \tag{86}$$

and note that $\delta_m \to \delta$. We claim $\delta = \rho$ by showing δ also is the solution to φ. Consider

$$\varphi(\delta_m) = \sum_{k\geq 0} p_k \left(\delta_m\right)^k \tag{87}$$
$$= \sum_{k\geq 0} \left(\mathbb{P}\left(Z_m/\mu^m = 0\right)\right)^k \mathbb{P}(Z_1 = k) \tag{88}$$
$$= \sum_{k\geq 0} \mathbb{P}\left(Z_{m+1}/\mu^{m+1} = 0 \mid Z_1 = k\right) \mathbb{P}(Z_1 = k) \tag{89}$$
$$= \mathbb{P}\left(Z_{m+1}/\mu^{m+1} = 0\right) = \delta_{m+1} \tag{90}$$

Take limit of both sides, note that φ is continuous by construction, we get

$$\varphi(\delta) = \delta \implies \delta = \rho \implies \mathbb{P}\left(\lim Z_n/\mu^n = 0\right) = \rho \tag{91}$$

Furthermore, by the assumption that $\delta < 1$, we have

$$1 - \delta = 1 - \rho \tag{92}$$

so that

$$\mathbb{P}\left(\lim Z_n/\mu^n > 0\right) = \mathbb{P}\left(Z_n > 0 \text{ for some n}\right) \tag{93}$$

as desired.

□

Problem 4.3.12.

Proof. First, we show ρ^{Z_n} is a martingale. It suffices to show that

$$\mathbb{E}\left(\rho^{Z_{n+1}} \mid \mathcal{F}_n\right) = \rho^{Z_n} \tag{94}$$

Observe that

$$\mathbb{E}\left(\rho^{Z_{n+1}} \mid \mathcal{F}_n\right) = \mathbb{E}\left(\rho^{Z_{n+1}} \mid Z_n\right) = \mathbb{E}\left(\rho^{\xi_1^{n+1}+\cdots+\xi_{Z_n}^{n+1}} \mid Z_n\right) \tag{95}$$

then on the event that $\{Z_n = k\}$, we have

$$\mathbb{E}\left(\rho^{\xi_1^{n+1}+\cdots+\xi_{Z_n}^{n+1}} \mid Z_n\right)\mathbf{1}_{Z_n=k} = \mathbb{E}\left(\rho^{\xi_1^{n+1}+\cdots+\xi_k^{n+1}}\right) \tag{96}$$
$$= \left(\mathbb{E}\rho^{\xi_1^{n+1}}\right)^k \tag{97}$$
$$= \left(\sum_{i=0}^{\infty} p_i \rho^i\right)^k = \varphi(\rho)^k = \rho^k \tag{98}$$

Note that the above holds for all $\{Z_n = k\}$, so that we have $\mathbb{E}\left(\rho^{Z_{n+1}} \mid \mathcal{F}_n\right) = \rho^{Z_n}$, which implies ρ^{Z_n} is a martingale.

For the second part, on the event $\{Z_0 = x\}$, by the property of martingale, we have

$$\mathbb{E}\left(\rho^{Z_n} \mid Z_0 = x\right) = \rho^x \tag{99}$$

Furthermore, since $\lim Z_n$ is either zero or infinity, i.e.,

$$\mathbb{P}\left(\lim Z_n = 0\right) + \mathbb{P}\left(\lim Z_n = \infty\right) = 1 \tag{100}$$

and $\rho < 1 \implies \rho^\infty = 0$, then

$$\begin{aligned}
\rho^x = \lim_{n\to\infty} \mathbb{E}\left(\rho^{Z_n} \mid Z_0 = x\right) &= \mathbb{E}\left(\lim_{n\to\infty} \rho^{Z_n} \mid Z_0 = x\right) \quad \text{by BCT} &(101)\\
&= \rho^0 \cdot \mathbb{P}\left(\lim Z_n = 0 \mid Z_0 = x\right) + \rho^\infty \cdot \mathbb{P}\left(\lim Z_n = \infty \mid Z_0 = x\right) &(102)\\
&= \mathbb{P}\left(\lim Z_n = 0 \mid Z_0 = x\right) &(103)\\
&= \mathbb{P}\left(Z_n = 0 \text{ for some } n \geq 1 \mid Z_0 = x\right) &(104)
\end{aligned}$$

as desired. □

Problem 4.4.2.

Proof. Suppose X_n is a sub-martingale and $M \leq N$ with $\mathbb{P}(N \leq k) = 1$. If $M = N$, then we are done. Assume $M < N$, then we may define a predictable

$$K_n = \mathbf{1}_{M<n\leq N} \tag{105}$$

It follows that

$$\left(K \cdot X\right)_n = X_{N\wedge n} - X_{M\wedge n} \tag{106}$$

is a sub-martingale. Thus, we have

$$\mathbb{E}\left(K \cdot X\right)_0 \leq \mathbb{E}\left(K \cdot X\right)_k \implies 0 \leq \mathbb{E}X_{N\wedge k} - \mathbb{E}X_{M\wedge k} \implies \mathbb{E}X_M \leq \mathbb{E}X_N \tag{107}$$

□

Problem 4.4.3.

Proof. We first show $A \in \mathcal{F}_N$. For fixed $n \in \mathbb{N}$, observe that

$$\begin{aligned}
\{N \leq n\} \cap A &= \{M \leq N \leq n\} \cap A &(108)\\
&= \{N \leq n\} \cap \left(\{M \leq n\} \cap A\right) &(109)
\end{aligned}$$

Since M is a stopping time, then by definition

$$\{M \leq n\} \cap A \in \mathcal{F}_n \tag{110}$$

Combined with the fact N is a stopping time, we know

$$\{N \leq n\} \cap A \in \mathcal{F}_n \implies A \in \mathcal{F}_N \tag{111}$$

and $A^c \in \mathcal{F}_N$. By the above result, we have

$$\left.\begin{array}{l}\{M \leq n\} \cap A \in \mathcal{F}_n\\ \{N \leq n\} \cap A^c \in \mathcal{F}_n\end{array}\right\} \implies \{L \leq n\} \in F_n \tag{112}$$

which proves the assertion. □

Problem 4.4.4.

Proof. Assume the same setting as above. Note that $L \leq N$. By exercise 4.4.2, we have

$$\mathbb{E}X_L \leq \mathbb{E}X_N \tag{113}$$

Breaking the LHS into two parts, we have

$$\mathbb{E}X_L = \mathbb{E}X_L \mathbf{1}_A + \mathbb{E}X_L \mathbf{1}_{A^c} = \mathbb{E}X_M \mathbf{1}_A + \mathbb{E}X_N \mathbf{1}_{A^c} \tag{114}$$

then

$$\mathbb{E}X_N \left(\mathbf{1}_A + \mathbf{1}_{A^c}\right) \geq \mathbb{E}X_M \mathbf{1}_A + \mathbb{E}X_N \mathbf{1}_{A^c} \tag{115}$$

$$\implies \mathbb{E}X_N \mathbf{1}_A \geq \mathbb{E}X_M \mathbf{1}_A \tag{116}$$

Since the choice of $A \in \mathcal{F}_M$ is arbitrary and $X_M \in F_M$, then we get

$$\mathbb{E}\left[X_N \mid \mathcal{F}_M\right] \geq \mathbb{E}\left[X_M \mid \mathcal{F}_M\right] = X_M \tag{117}$$

as desired. □

Problem 4.4.10.

Proof. We use L^p convergence theorem to prove the assertion. It suffices to argue

$$\mathbb{E}|X_n|^2 = \mathbb{E}X_n^2 < \infty \tag{118}$$

Consider

$$\mathbb{E}X_n^2 = \mathbb{E}\left(X_0 + \xi_1 + \cdots \xi_n\right)^2 \tag{119}$$

$$= \mathbb{E}\left(X_0 + \sum_{k=1}^n \xi_k\right)^2 \tag{120}$$

$$= \mathbb{E}\left(X_0^2 + 2X_0 \sum_{k=1}^n \xi_k + \left(\sum_{k=1}^n \xi_k\right)^2\right) \tag{121}$$

$$= \mathbb{E}X_0^2 + 2\mathbb{E}X_0 \sum_{k=1}^n \xi_k + \mathbb{E}\left(\sum_{k=1}^n \xi_k\right)^2 \tag{122}$$

$$= \mathbb{E}X_0^2 + 2\mathbb{E}X_0 \sum_{k=1}^n \xi_k + \mathbb{E}\left(\sum_{k=1}^n \xi_k^2 + 2\sum_{j<k<n} \xi_j \xi_k\right) \tag{123}$$

$$= \mathbb{E}X_0^2 + \mathbb{E}\sum_{k=1}^n \xi_k^2 + \left(2\sum_{k=1}^n \mathbb{E}X_0 \xi_k + 2\sum_{j<k<n} \mathbb{E}\xi_j \xi_k\right) \tag{124}$$

By orthogonality of martingale increments, the term in parentheses equal zero, so

$$\mathbb{E}X_n^2 = \mathbb{E}X_0^2 + \mathbb{E}\sum_{k=1}^n \xi_k^2 < \infty \tag{125}$$

as desired. □

Problem 4.6.2.

Proof. We show X_n on $I_{k,n}$ defines a martingale. Fix $I_{k,n} \in \mathcal{F}_n$ arbitrary, observe that if

$$\mathbb{E}X_{n+1}\mathbf{1}_{I_{k,n}} = \mathbb{E}X_n\mathbf{1}_{I_{k,n}} \tag{126}$$

then we must have

$$\int_{I_{k,n}} X_{n+1} = \int_{I_{k,n}} X_n \tag{127}$$

Since our choice of $I_{k,n}$ is arbitrary and any event in $\mathcal{F}_n$ are of the form $I_{k,n}$, then it follows from the definition of conditional expectation that

$$\mathbb{E}\left[X_{n+1} \mid \mathcal{F}_n\right] = \mathbb{E}\left[X_n \mid \mathcal{F}_n\right] = X_n \tag{128}$$

Thus, for the first claim, it suffices to show equation 126. Note that by construction, $\{X_n\}$'s are constant given k, n, then

$$\mathbb{E}X_n\mathbf{1}_{I_{k,n}} = X_n\mathbb{P}\left(I_{k,n}\right) = \frac{f(\frac{k+1}{2^n}) - f(\frac{k}{2^n})}{\frac{1}{2^n}} \cdot \frac{1}{2^n} \tag{129}$$

$$= f\left(\frac{k+1}{2^n}\right) - f\left(\frac{k}{2^n}\right) \tag{130}$$

For the other side, observe the following trickery

$$\frac{k}{2^n} = \frac{2k}{2^{n+1}} \tag{131}$$

$$\frac{k+1}{2^n} = \frac{2k+2}{2^{n+1}} \tag{132}$$

then

$$\mathbb{E}X_{n+1}\mathbf{1}_{I_{k,n}} = \mathbb{E}X_{n+1}\mathbf{1}_{I_{2k,n+1}} + \mathbb{E}X_{n+1}\mathbf{1}_{I_{2k+1,n+1}} \tag{133}$$

$$= f\left(\frac{2k+2}{2^{n+1}}\right) - f\left(\frac{2k}{2^{n+1}}\right) \tag{134}$$

which proves that X_n is a martingale on $I_{k,n}$.

Since we have shown X_n is a martingale, then to show convergence in L^1 and a.s., it suffices to show X_n is uniformly integrable by Theorem 4.6.7.

First, we exploit Lipschitz continutiy to get

$$|X_n| = \left|\frac{f(\frac{k+1}{2^n}) - f(\frac{k}{2^n})}{\frac{1}{2^n}}\right| \le \left|\frac{K \cdot \frac{1}{2^n}}{\frac{1}{2^n}}\right| \le K \tag{135}$$

for all $n \in \mathbb{N}$. It follows that for sufficiently large M with $M > K$, $\{|X_i| > M\}$ is a measure zero set. Consequently, we have

$$\lim_{M\to\infty}\left(\sup_{i\in I}\mathbb{E}\left(|X_i|; |X_i| > M\right)\right) = 0 \tag{136}$$

It immediately follows that X_n converges a.s. and in L^1 to X_∞.

Lastly, we prove the result regarding integrals. Since we have a.s. convergence and $|X_n| \le K$ and $\mathcal{F}_n \uparrow \mathcal{F}_\infty$, then we may apply DCT for conditional expectations and get

$$\mathbb{E}\left[X_n \mid \mathcal{F}_n\right] \to \mathbb{E}\left[X_\infty \mid \mathcal{F}_\infty\right] \tag{137}$$

Since the above as random varible are also bounded by the fact that Lipschitz bound (equation 135) holds for all $n \in \mathbb{N}$, then BCT is justified. Combined with the definition of conditional expectation, we get

$$\int_{I_{k,n}} X_n = \int_{I_{k,n}} \mathbb{E}\left[X_n \mid \mathcal{F}_n\right] \to \int_{I_{k,n}} \mathbb{E}\left[X_\infty \mid \mathcal{F}_\infty\right] = \int_{I_{k,n}} X_\infty \tag{138}$$

for fixed $I_{k,n}$. Denote a_k, b_k as the end point of $I_{k,n}$, then

$$\int_{I_{k,n}} X_n = X_n \int \mathbf{1}_{I_{k,n}} = \frac{f(\frac{k+1}{2^n}) - f(\frac{k}{2^n})}{\frac{1}{2^n}} \cdot \frac{1}{2^n} \tag{139}$$

$$= f(b_k) - f(a_k) \tag{140}$$

$$= \int_{a_k}^{b_k} X_\infty \tag{141}$$

Since f is continuous, for any a, b, we may send $a_k \to a, b_k \to b$, which completes the proof. □

Problem 4.6.3.

Proof. Note that $\mathcal{F}_n \uparrow \mathcal{F}_\infty$, then by Theorem 4.6.8, we have that

$$\mathbb{E}\left[f \mid F_n\right] \to \mathbb{E}\left[f \mid F_\infty\right] \tag{142}$$

almost surely and in L^1. Since $f \in \mathcal{F}_\infty$, then we have

$$\mathbb{E}\left[f \mid F_n\right] \to \mathbb{E}\left[f \mid F_\infty\right] = f \tag{143}$$

in L^1. □

Problem 4.6.4.

Proof. First, we make some observations. Note that

$$\mathbb{P}\left(\{\lim_{n\to\infty} X_n = \infty\}^c \cup \{\lim_{n\to\infty} X_n = \infty\}\right) = 1 \tag{144}$$

Thus, to prove the claim, it suffices to show

$$\{\lim_{n\to\infty} X_n = \infty\}^c \subseteq D \tag{145}$$

For the set up, denote

$$\mathcal{F}_n = \sigma\left(X_1, \cdots, X_n\right) \text{ and } \mathcal{F}_\infty = \sigma\left(\cup_n^\infty X_n\right) \tag{146}$$

By construction, we have $\mathcal{F}_n \uparrow \mathcal{F}_\infty$ and $D \in F_\infty$, then Levy 0-1 law implies that

$$\mathbb{E}\left(\mathbf{1}_D \mid F_n\right) = \mathbb{P}\left(D \mid F_n\right) \to \mathbf{1}_D \text{ a.s.} \tag{147}$$

Consequently, for fixed positive x, take $\omega \in \{X_n \leq x \text{ i.o.}\}$. Note that if $X_n \leq x$ infinitely often, then there must exists some further sequences of index $\{n_j\}$ beyond a threshold that ensures $X_{n_j} \leq x$ for all n_j. It follows that

$$\mathbf{1}_D(\omega) = \lim_{n\to\infty} \mathbb{P}\left(D \mid F_n\right) = \lim_{j\to\infty} \mathbb{P}\left(D \mid F_{n_j}\right) \geq \delta(x) > 0 \tag{148}$$

which implies that

$$\mathbf{1}_D(\omega) = 1 \implies \omega \in D \implies \{X_n \leq x \text{ i.o.}\} \subseteq D \tag{149}$$

Since the above inclusion holds for every $x \in \mathbb{N}$, then we must have

$$\bigcup_{x\in\mathbb{N}} \{X_n \leq x \text{ i.o.}\} \subseteq D \tag{150}$$

Lastly, we claim that

$$\bigcup_{x\in\mathbb{N}} \{X_n \leq x \text{ i.o.}\} = \{\lim_{n\to\infty} X_n = \infty\}^c \tag{151}$$

Note that for any $x, X_n \leq x$ infinitely often implies that after a certain threshold, $X_n \neq \infty$, so that

$$\bigcup_{x\in\mathbb{N}} \{X_n \leq x \text{ i.o.}\} \subseteq \{\lim_{n\to\infty} X_n = \infty\}^c \tag{152}$$

If $\lim_{n\to\infty} X_n = \infty$ is not true, then for index n after any threshold, $X_n \neq \infty$. Which implies that for all threshold, X_n is bounded infinitely often. Thus, we have

$$\{\lim_{n\to\infty} X_n = \infty\}^c \subseteq \bigcup_{x\in\mathbb{N}} \{X_n \leq x \text{ i.o.}\} \tag{153}$$

which proves the claim. □

Problem 4.6.5.

Proof. Assume $p_0 = \mathbb{P}(\xi = 0) > 0$. Denote

$$D = \{\lim_{n\to\infty} Z_n = 0\} \tag{154}$$

and $\mathcal{F}_n = \sigma(Z_1, \cdots, Z_n)$ in which $\mathcal{F}_n \uparrow \mathcal{F}_\infty$. Before using 4.6.4, we need to justify its assumption.

We need to find such $\delta(x) > 0$. Fix positive x. Observe that for the event $\{Z_n \leq x\}$, there are two cases: (i) $Z_n \neq 0$ and (ii) $Z_n = 0$. Consider

1. In this case, on the event $\{0 < Z_n \leq x\}$, we have

$$\mathbb{P}(D \mid F_n) \geq p_0^{Z_n} \geq p_0^x > 0 \tag{155}$$

2. If $Z_n = 0$, then we trivially have that

$$\mathbb{P}(D \mid F_n) = 1 \geq p_0^x > 0 \tag{156}$$

Thus, we may conclude on $\{X_n \leq x\}$

$$\mathbb{P}(D \mid F_n) \geq \delta(x) := p_0^x > 0 \tag{157}$$

so that from 4.6.4

$$\mathbb{P}\left(D \cup \{\lim_{n\to\infty} Z_n = \infty\}\right) = \mathbb{P}\left(\{\lim_{n\to\infty} Z_n = 0\} \cup \{\lim_{n\to\infty} Z_n = \infty\}\right) \tag{158}$$

$$= \mathbb{P}\left(\lim_{n\to\infty} Z_n = 0 \text{ or } \infty\right) = 1 \tag{159}$$

as desired. □

Problem 4.8.3.

Proof. Note that if $\mathbb{E}T = \infty$, then problem is trivial. For the same reason, we assume that $T < \infty$ and $\mathbb{E}T < \infty$. For the set up, we denote

$$X_n := S_n^2 - n\sigma^2 \tag{160}$$

Now, we justify why we may use Theorem 4.8.2. Consider

$$\begin{aligned}
\mathbb{E}|X_T| &= \mathbb{E}\left|S_T^2 - T\sigma^2\right| &(161)\\
&\le \mathbb{E}\left|S_T^2\right| &(162)\\
&= \mathbb{E}\left[S_{T-1} + \xi\right]^2 &(163)\\
&\le \mathbb{E}\left(a + \xi\right)^2 &(164)\\
&= \mathrm{Var}(a+\xi) + \left(\mathbb{E}(a+\xi)\right)^2 &(165)\\
&\le \sigma^2 + a^2 < \infty &(166)
\end{aligned}$$

Then, we show $X_n \mathbf{1}_{T>n}$ is uniformly integrable. Observe that

$$|X_n \mathbf{1}_{T>n}| = \left|\left(S_n^2 - n\sigma^2\right)\mathbf{1}_{T>n}\right| \le a^2 \tag{167}$$

Clearly, for all $\epsilon > 0$, the set

$$\{|X_n \mathbf{1}_{T>n}| > a^2 + \epsilon\} \tag{168}$$

is of measure zero. Thus, we get uniform integrablity, so that Theorem 4.8.2 implies $X_{T\wedge n}$ is uniform integrable.

Consequently, by OST from class, we have

$$\begin{aligned}
\mathbb{E}X_T = \mathbb{E}X_0 = 0 &\implies \mathbb{E}S_T^2 = \sigma^2 \mathbb{E}T &(169)\\
&\implies \mathbb{E}T = \frac{\mathbb{E}S_T^2}{\sigma^2} \ge \frac{a^2}{\sigma^2} &(170)
\end{aligned}$$

as desired. □

Problem 4.8.4.

Proof. We use the same notation as above and assume that $\mathbb{E}T < \infty$. Note that $X_{T\wedge n}$ is also a martiangle, then we must have

$$\mathbb{E}X_{T\wedge n} = \mathbb{E}X_0 = 0 \tag{171}$$

which implies

$$\mathbb{E}S_{T\wedge n}^2 = \sigma^2 \mathbb{E}\left[T \wedge n\right] \tag{172}$$

Since $0 \le T \wedge n \uparrow T$, we may apply MCT and deduce that

$$\sigma^2 \mathbb{E}\left[T \wedge n\right] \to \sigma^2 \mathbb{E}T \tag{173}$$

that is

$$\mathbb{E}S_{T\wedge n}^2 \to \sigma^2 \mathbb{E}T < \infty \tag{174}$$

which must holds true for all n and particularly for sup over n. Thus, L^p convergence theorem applies, so we may deduce that

$$S_{T\wedge n} \to S_T \tag{175}$$

almost surely and in L^2. Chain the above result, we must have

$$\mathbb{E}S_T^2 = \sigma^2 \mathbb{E}T \tag{176}$$

□

Problem 4.8.5.

Proof. (a) For the set up, we assume $S_0 = x$ and denote

$$Z_n = (S_n - (p-q)n)^2 - n(1-(p-q)^2) \tag{177}$$

By assumption, we know $\mathbb{E}V_0$ is finite. If $\mathbb{E}V_0^2 = \infty$, then the result is trivial. Thus, we must have $\mathbb{E}V_0^2 < \infty$.

Now, we show $\mathbb{E}Z_{V_0} = \mathbb{E}Z_0$ using theorem 4.8.2. There are two conditions to verify. Consider

$$\mathbb{E}|Z_{V_0}| = \mathbb{E}\Big|(S_{V_0} - (p-q)V_0)^2 - V_0(1-(p-q)^2)\Big| \tag{178}$$

$$\leq \mathbb{E}V_0^2 + \mathbb{E}V_0 < \infty \tag{179}$$

For the second condition, note the fact that before hitting stopping time V_0, gambler's wealth is bounded. Intuitively, that is if the gambler wins money in every single round before time V_0, then

$$S_n \mathbf{1}_{V_0>n} \leq x + n < \infty \tag{180}$$

Therefore, we have

$$|Z_n \mathbf{1}_{V_0>n}| = \Big|(S_n - (p-q)n)^2 - n(1-(p-q)^2)\Big|\mathbf{1}_{V_0>n} \tag{181}$$

$$\leq (S_n + n)^2 \mathbf{1}_{V_0>n} + n\mathbf{1}_{V_0>n} < \infty \tag{182}$$

The above immediately implies that $Z_n \mathbf{1}_{V_0>n}$ is uniformly integrable, so that Theorem 4.8.2 is justified and

$$\mathbb{E}Z_{V_0} = \mathbb{E}Z_0 = x^2 \tag{183}$$

then

$$(p-q)^2 \mathbb{E}V_0^2 - (1-(p-q)^2)\mathbb{E}V_0 = x^2 \tag{184}$$

Write $q = 1-p$ and solve for $\mathbb{E}V_0^2$, we get

$$\mathbb{E}V_0^2 = \frac{4p^2x + 2px^2 - 4px - x^2}{(2p-1)^3} \tag{185}$$

It follows that

$$\mathrm{Var}V_0 = \mathbb{E}V_0^2 - (\mathbb{E}V_0)^2 = \frac{4p^2x + 2px^2 - 4px - x^2}{(2p-1)^3} - \frac{x^2}{(1-2p)^2} \tag{186}$$

$$= \frac{4(p-1)px}{(2p-1)^3} \tag{187}$$

$$= x \cdot \frac{1-(p-q)^2}{(q-p)^3} \tag{188}$$

which proves the claim.

(b) For the set up, we denote

$$V_y = \min\{n \geq 0 : S_n = y\} \tag{189}$$

and

$$N_y = V_{y-1} - V_y \tag{190}$$

for $y \in \{1, \cdots, x-1, x\}$. By construction, we know N_y's must be IID with finite variance, say $\mathrm{Var}N_1 := c$. Since we assume $S_0 = x$, then we must have

$$V_x = \min\{n \geq 0 : S_n = x\} = 0 \tag{191}$$

Thus, we get

$$V_0 = V_0 + V_x = \sum_{y=1}^{x} N_y \tag{192}$$

so that

$$\mathrm{Var}V_0 = \sum_{y=1}^{x} \mathrm{Var}N_y = x\mathrm{Var}N_1 = cx \tag{193}$$

as desired.

□

Problem 4.8.6.

Proof. (a) For the set up, we denote the exponential martingale as

$$X_n := \exp(\theta S_n)/\phi(\theta)^n \tag{194}$$

where

$$\phi(\theta) = \mathbb{E}\exp(\theta\xi_i) = pe^{\theta} + qe^{-\theta} \tag{195}$$

First, we make some observation regarding function ϕ. At $\theta = 0$, we have

$$\phi(0) = 1 \tag{196}$$
$$\phi'(0) = p - q < 0 \tag{197}$$
$$\phi''(\theta) = \phi(\theta) > 0 \tag{198}$$

Therefore, assume $\theta \leq 0$, we must have

$$\phi(\theta) \geq 1 \tag{199}$$

Now, we justify why we may use Theorem 4.8.2, consider

$$\mathbb{E}|X_{V_0}| = \mathbb{E}\left[\frac{e^{\theta S_{V_0}}}{\phi(\theta)^{V_0}}\right] \leq \mathbb{E}\left[e^{\theta S_{V_0}}\right] = 1 < \infty \tag{200}$$

Since $S_0 = x > 0$, then S_n must be positive before it hit zero, that implies $\theta S_n < 0$ and

$$|X_n \mathbf{1}_{V_0 > n}| = |\exp(\theta S_n)/\phi(\theta)^n| \mathbf{1}_{V_0 > n} \tag{201}$$
$$\leq \exp(\theta S_n)\mathbf{1}_{V_0 > n} \tag{202}$$
$$\leq \exp(\theta S_n) \leq 1 \tag{203}$$

Uniform integrability immediately follows, then we know $X_n\mathbf{1}_{V_0>n}$ is uniformly integrable. Thus, we may use theorem 4.8.2 to conclude

$$\mathbb{E}X_{V_0} = \mathbb{E}X_0 = e^{x\theta} \tag{204}$$

that is

$$\mathbb{E}\left[\frac{e^{\theta S_{V_0}}}{\phi(\theta)^{V_0}}\right] = \mathbb{E}\left[\phi(\theta)^{-V_0}\right] = e^{x\theta} \tag{205}$$

(b) Suppose $0 < s < 1$, then we have

$$\phi(\theta) = pe^{\theta} + qe^{-\theta} = \frac{1}{s} \tag{206}$$

and

$$\phi(\theta)^{-1} = s \tag{207}$$

In this notation, part (a) translates into

$$\mathbb{E}s^{V_0} = e^{x\theta} \tag{208}$$

Solve for e^{θ} from equation 206, we get

$$e^{\theta} = \frac{1 - \sqrt{1 - 4pqs^2}}{2ps} \text{ or } \frac{1 + \sqrt{1 - 4pqs^2}}{2ps} \tag{209}$$

Note that since we assume $\theta \leq 0$, then

$$e^{\theta} \leq 1 \tag{210}$$

However, the second root is at least one, i.e.,

$$\frac{1 + \sqrt{1 - 4pqs^2}}{2ps} \geq \frac{1}{2ps} \geq \frac{1}{s} > 1 \tag{211}$$

Thus, the second root is eliminated by this criteria, and we have

$$\mathbb{E}s^{V_0} = \left(\frac{1 - \sqrt{1 - 4pqs^2}}{2ps}\right)^x \tag{212}$$

as desired.

(c) Note that in problem 2, part (b), we have defined

$$N_y = V_{y-1} - V_y \tag{213}$$

for $y \in \{1, \cdots, x-1, x\}$ and

$$V_0 = \sum_{y=1}^{x} N_y \tag{214}$$

Thus, we have

$$\mathbb{E}s^{V_0} = \mathbb{E}\left[s^{\sum_{y=1}^{x} N_y}\right] \tag{215}$$

$$= \Pi_{y=1}^{x} \mathbb{E}s^{N_y} \tag{216}$$

$$= \left(\mathbb{E}s^{N_1}\right)^x \tag{217}$$

as desired.

□

Problem 4.8.7.

Proof. First, note that $X_n := S_n^2 - n$ is martingale.

If $\mathbb{E}T = \infty$, then we are done since Cauchy-Schwarz inequality tells us

$$\infty = (\mathbb{E}[T \cdot 1]) \leq \mathbb{E}T^2 \cdot \mathbb{E}\left[1^2\right] = \mathbb{E}T^2 \tag{218}$$

Therefore, we assume $\mathbb{E}T < \infty$.

For the set up, we use Theorem 4.8.2 to get desired result. First, we need to justify the assumption of Theorem 4.8.2. Consider

$$\begin{aligned} \mathbb{E}|X_T| &= \mathbb{E}\left|S_T^2 - T\right| & (219)\\ &\leq \mathbb{E}S_T^2 + \mathbb{E}T & (220)\\ &\leq a^2 + \mathbb{E}T < \infty & (221) \end{aligned}$$

and

$$\begin{aligned} |X_n \mathbf{1}_{T>n}| &= \left|S_n^2 - n\right| \mathbf{1}_{T>n} & (222)\\ &\leq S_n^2 \mathbf{1}_{T>n} + n\mathbf{1}_{T>n} & (223)\\ &\leq a^2 + T < \infty & (224) \end{aligned}$$

Thus, uniform integrability of $X_n \mathbf{1}_{T>n}$ immediately follows. Theorem 4.8.2 implies that $X_{n\wedge T}$ is uniformly integrable. Hence,

$$0 = \mathbb{E}X_0 = \mathbb{E}X_T \implies \mathbb{E}S_T^2 = \mathbb{E}[T] \tag{225}$$

By definition of stopping time T, we know $S_T \leq a$, then

$$a^2 = \mathbb{E}T \tag{226}$$

Now, note that Y_n is a martingale iff

$$\mathbb{E}[Y_{n+1} \mid \mathcal{F}_n] - Y_n = b + c - 5 + (2b - 6) \cdot n = 0 \tag{227}$$

Solve for the above, we get $b = 3, c = 2$, then

$$Y_n = S_n^4 - 6nS_n^2 + 3n^2 + 2n \tag{228}$$

is a martingale. If Y_n is a martingale, then so is $Y_{n\wedge T}$. By property of martingale, we get

$$\mathbb{E}Y_{n\wedge T} = \mathbb{E}Y_0 = 0 \tag{229}$$

that is

$$\mathbb{E}\left[S_{n\wedge T}^4 + 3(n \wedge T)^2 + 2(n \wedge T)\right] = \mathbb{E}\left[6(n \wedge T) S_{n\wedge T}^2\right] \tag{230}$$

By the same tautology of stopping time definition and B/MCT, we get

$$\mathbb{E}\left[a^4 + 3T^2 + 2T\right] = \mathbb{E}\left[6Ta^2\right] \tag{231}$$

Solve the above, we get

$$\mathbb{E}T^2 = \frac{5a^4 - 2a^2}{3} \tag{232}$$

as desired. □

Problem 7.1.3.

Proof. If $\mathbb{E}W$ or $\mathbb{E}W^2 = \infty$, then we are done. Assume not, then $W < \infty$ almost surely. Note that the fact $B_s \sim N(0, s)$ implies

$$\mathbb{E}B_s = 0 \text{ and } \mathbb{E}B_s^2 = \text{Var}B_s = t \tag{233}$$

Note that $\mathbb{E}B_s = 0 \implies \mathbb{E}|B_s| < \infty$, so Fubini is justfied. Apply Fubini's theorem, we get

$$\mathbb{E}W = \mathbb{E}\left[\int_0^t B_s ds\right] = \int_0^t \mathbb{E}B_s ds = 0 \tag{234}$$

Cauchy-Schwarz tells us

$$\mathbb{E}|B_sB_t| \leq \sqrt{\mathbb{E}B_s^2} \cdot \sqrt{\mathbb{E}B_t^2} = (st)^{1/2} < \infty \tag{235}$$

so that Fubini is justified for the following computation. Consider

$$\mathbb{E}W^2 = \mathbb{E}\left(\int_0^t B_s ds\right)^2 = \mathbb{E}\left(\int_0^t B_{s_1} ds_1 \cdot \int_0^t B_{s_2} ds_2\right) \tag{236}$$

$$= \mathbb{E}\left(\int_0^t \int_0^t B_{s_1} B_{s_2} ds_1 ds_2\right) \tag{237}$$

$$= \int_0^t \int_0^t \mathbb{E}\left[B_{s_1} B_{s_2}\right] ds_1 ds_2 \tag{238}$$

$$= \int_0^t \left(\int_0^t s_1 \wedge s_2 ds_1\right) ds_2 \tag{239}$$

$$= \int_0^t \left(\int_0^{s_2} s_1 \wedge s_2 ds_1 + \int_{s_2}^t s_1 \wedge s_2 ds_1\right) ds_2 \tag{240}$$

$$= \int_0^t \left(\int_0^{s_2} s_1 ds_1 + \int_{s_2}^t s_2 ds_1\right) ds_2 \tag{241}$$

$$= \int_0^t \left({s_2}^2/2 + s_2 t - s_2^2\right) ds_2 \tag{242}$$

$$= t \cdot \frac{t^2}{2} - \frac{1}{2} \cdot \frac{t^3}{3} = \frac{t^3}{3} \tag{243}$$

We claim W must be Gaussian. Apply integration by parts on W, we get

$$W = \int_0^t B_s ds = tB_t - \int_0^t s dB_s \tag{244}$$

$$= \int_0^t t dB_s - \int_0^t s dB_s \tag{245}$$

$$= \int_0^t (t - s) dB_s \tag{246}$$

Denote the n-th partition of $[0, t]$ as

$$\mathcal{P}_n = \{0 = t_0 < t_1 < \cdots < t_i < \cdots < t_n = t\} \tag{247}$$

then the integral becomes

$$\int_0^t (t - s) dB_s = \lim_{n \to \infty} \sum_{i=1}^n (t_i - s)\left(B_{t_i} - B_{t_{i-1}}\right) \tag{248}$$

For all partition, by independence of increment, we know family of $\{\left(B_{t_i} - B_{t_{i-1}}\right)\}_{i \in \mathbb{N}}$ is independently Gaussian. Therefore, W, as the limit of sum of independent Gaussians, is also Gaussian. Combined with our calculation, we know $W \sim N(0, t^3/3)$, as desired. ∎

Problem 7.1.6.

Proof. Observe that

$$\mathbb{E}\left(\sum_{m\leq 2^n}\Delta_{m,n}^2 - t\right)^2 = \text{Var}\left(\sum_{m\leq 2^n}\Delta_{m,n}^2 - t\right) + \left(\mathbb{E}\left[\sum_{m\leq 2^n}\Delta_{m,n}^2 - t\right]\right)^2 \tag{249}$$

$$= \sum_{m\leq 2^n}\text{Var}\Delta_{m,n}^2 + \left(\sum_{m\leq 2^n}\mathbb{E}\Delta_{m,n}^2 - t\right)^2 \tag{250}$$

Thus, it suffices to find $\text{Var}\Delta_{m,n}^2$ and $\mathbb{E}\Delta_{m,n}^2$. Note that by property of Brownian motion, we have

$$\Delta_{m,n} = B(tm2^{-n}) - B(t(m-1)2^{-n}) \tag{251}$$

$$\stackrel{d}{=} 2^{-n/2}\left(B(tm) - B(t(m-1))\right) \tag{252}$$

$$\stackrel{d}{=} 2^{-n/2}\left(B(t) - B(0)\right) \tag{253}$$

$$= 2^{-n/2}\Delta_{1,0} \tag{254}$$

By definition, we know $\Delta_{1,0} \stackrel{d}{=} N(0,t)$. It follows that

$$\mathbb{E}\Delta_{m,n}^2 = 2^{-n}\mathbb{E}\Delta_{1,0}^2 = 2^{-n}\cdot t \tag{255}$$

and

$$\text{Var}\Delta_{m,n}^2 = 2^{-2n}\text{Var}\Delta_{1,0}^2 = 2^{-2n}\cdot 2\left(\text{Var}\Delta_{1,0}\right)^2 = 2^{-2n+1}\cdot t^2 \tag{256}$$

Plug the above back to equation 250, we get

$$\sum_{m\leq 2^n}\text{Var}\Delta_{m,n}^2 + \left(\sum_{m\leq 2^n}\mathbb{E}\Delta_{m,n}^2 - t\right)^2 = 2^n\cdot\left(2^{-2n+1}\cdot t^2\right) + 0^2 = 2^{-n+1}\cdot t^2 \tag{257}$$

Thus, we may conclude

$$\mathbb{E}\left(\sum_{m\leq 2^n}\Delta_{m,n}^2 - t\right)^2 = 2^{-n+1}\cdot t^2 \tag{258}$$

For the second part, applying Markov's inequality, we get

$$\frac{1}{n^2}\cdot\mathbb{P}\left(\left|\sum_{m\leq 2^n}\Delta_{m,n}^2 - t\right| \geq \frac{1}{n}\right) \leq \mathbb{E}\left(\sum_{m\leq 2^n}\Delta_{m,n}^2 - t\right)^2 \tag{259}$$

Plug in the value we computed, we get

$$\mathbb{P}\left(\left|\sum_{m\leq 2^n}\Delta_{m,n}^2 - t\right| \geq \frac{1}{n}\right) \leq 2^{-n+1}\cdot n^2t^2 \tag{260}$$

Observe the simple fact that

$$\sum_{n\to\infty} 2^{-n+1}\cdot n^2t^2 = 12t^2 < \infty \tag{261}$$

Thus, BC lemma implies that

$$\mathbb{P}\left(\left|\sum_{m\leq 2^n}\Delta_{m,n}^2 - t\right| \geq \frac{1}{n}\text{ i.o.}\right) = 0 \tag{262}$$

which proves the desired result. □

Problem 7.2.4.

Proof. (i) First, define the stopping time

$$\tau = \inf\{t \geq 0 : B(t) > 0\} \tag{263}$$

then theorem 7.2.4 implies that

$$\mathbb{P}(\tau = 0) = 1 \tag{264}$$

In other words, for sufficiently small $t > 0$, we must have $B(t) > 0$. Formally, that is

$$\limsup_{t \downarrow 0} B(t) > 0 \tag{265}$$

Also, continuity of Brownian motion ensures $B(t)$ is bounded. Combine with the fact that $f(t) > 0$ for all $t > 0$, we get

$$\frac{\limsup_{t \downarrow 0} B(t)}{f(t)} = \limsup_{t \downarrow 0} B(t)/f(t) = c \tag{266}$$

where $c \in [0, \infty]$ and is measurable to $\mathcal{F}_0^+$. By theorem 7.2.3, it follows that for any constant a, we have

$$\mathbb{P}_0(c = a) \in \{0, 1\} \tag{267}$$

which implies that c could only be a constant, almost surely.

(ii) We construct a new Brownian motion. By theorem 7.2.6, given Brownian motion $B(t)$ starts at zero, we know

$$X(t) = tB(1/t) \tag{268}$$

is also Brownian motion for $t > 0$. Then, define $s = \frac{1}{t}$, we apply theorem 7.2.8 on $X(t)$, which yields

$$\infty = \limsup_{t \to \infty} X(t)/\sqrt{t} = \limsup_{t \to \infty} \sqrt{t}B(1/t) = \limsup_{s \downarrow 0} \sqrt{1/s}B(s) = \limsup_{s \downarrow 0} \frac{B(s)}{\sqrt{s}} \tag{269}$$

with probability one, as desired.

□

Problem 7.2.2.

Proof. First, we make some observations. Note that $t \in (0, 1)$. Then, the event $\{L \leq t\}$ means the last time a Brownian motion visits zero is before time t. That is to say, between the time interval $(t, 1]$, the Brownian motion does not visit zero. In other words, this Brownian motion's zero hitting time must be after time one. If we use shift transformation to cut off the path before time t, so that time t became time zero, we get

$$\{L \leq t\} = \{T_0 \circ \theta_t > 1 - t\} \tag{270}$$

It follows directly from Theorem 7.2.1 that

$$\begin{aligned}
P_0(L \leq t) &= P_0(T_0 \circ \theta_t > 1 - t) && (271)\\
&= \mathbb{E}_0\left(\mathbf{1}_{T_0 \circ \theta_t > 1-t}\right) && (272)\\
&= \mathbb{E}_0\left(\mathbb{E}_0\left(\mathbf{1}_{T_0 \circ \theta_t > 1-t} \mid \mathcal{F}_t^+\right)\right) && (273)\\
&= \mathbb{E}_0\left(\mathbb{E}_0\left(\mathbf{1}_{T_0 > 1-t} \circ \theta_t \mid \mathcal{F}_t^+\right)\right) && (274)\\
&= \mathbb{E}_0\left(\mathbb{E}_{B_t}\left(\mathbf{1}_{T_0 > 1-t}\right)\right) && (275)\\
&= \mathbb{E}_0\left(\mathbb{P}_{B_t}(T_0 > 1 - t)\right) && (276)\\
&= \int \mathbb{P}(B_t = y \mid B_0 = 0)\, \mathbb{P}_y(T_0 > 1 - t)\, dy && (277)\\
&= \int \mathbb{P}_0(B_t = y)\, \mathbb{P}_y(T_0 > 1 - t)\, dy && (278)\\
&= \int p_t(0, y)\, \mathbb{P}_y(T_0 > 1 - t)\, dy && (279)
\end{aligned}$$

as desired. ☐

Problem 7.2.1.

Proof. First, we make some observation. Note that R is the first time that a Brownian motion hits zero after time one. If we use shift transformation to truncate the path before time one, so that time one became the starting time zero, we get

$$R = T_0 \circ \theta_1 + 1 \tag{280}$$

It follows directly from Theorem 7.2.1 that

$$\begin{aligned}
\mathbb{P}_x\left(R > 1+t\right) &= \mathbb{P}_x\left(T_0 \circ \theta_1 > t\right) && (281)\\
&= \mathbb{E}_x\left(\mathbf{1}_{T_0\circ\theta_1 > t}\right) && (282)\\
&= \mathbb{E}_x\left(\mathbf{1}_{T_0 > t} \circ \theta_1\right) && (283)\\
&= \mathbb{E}_x\left(\mathbb{E}_x\left(\mathbf{1}_{T_0 > t} \circ \theta_1\right) \mid \mathcal{F}_1^+\right) && (284)\\
&= \mathbb{E}_x\left(\mathbb{E}_{B_1}\left(\mathbf{1}_{T_0 > t} \circ \theta_1\right)\right) && (285)\\
&= \mathbb{E}_x\left(\mathbb{P}_{B_1}\left(T_0 > t\right)\right) && (286)\\
&= \int p_1(x,y)\mathbb{P}_y(T_0 > t)dy && (287)
\end{aligned}$$

as desired. ☐

Problem 7.4.2.

Proof. Run similar argument as in the proof of (7.2.3), we get

$$\mathbb{P}_0(R \le 1+t) = \int p_1(0,y)\mathbb{P}_y(T_0 \le t)dy = 2\int_0^\infty p_1(0,y)\mathbb{P}_y(T_0 \le t)dy \tag{288}$$

The second equality is due to the fact that Brownian motion is normally distributed, so that symmetry follows. Plug in the the density function (7.4.6) from example 7.4.2, we have

$$\begin{aligned}
\mathbb{P}_0(R \le 1+t) &= 2\int_0^\infty p_1(0,y)\mathbb{P}_y(T_0 \le t)dy && (289)\\
&= 2\int_0^\infty p_1(0,y)\mathbb{P}_0(T_y \le t)dy && (290)\\
&= 2\int_0^\infty (2\pi)^{-1/2}\exp\bigl(-y^2/2\bigr)\left(\int_0^t (2\pi s^3)^{-1/2} y \exp\bigl(-y^2/2s\bigr)ds\right)dy && (291)\\
&= \frac{1}{\pi}\int_0^\infty\int_0^t \exp\bigl(-y^2/2\bigr)s^{-3/2}y\exp\bigl(-y^2/2s\bigr)dsdy && (292)\\
&= \frac{1}{\pi}\int_0^\infty\int_0^t y\exp\bigl(-y^2/2 - y^2/2s\bigr)s^{-3/2}dsdy && (293)\\
&= \frac{1}{\pi}\int_0^t s^{-3/2}\left(\int_0^\infty y\exp\bigl(-y^2/2 - y^2/2s\bigr)dy\right)ds && (294)\\
&= \frac{1}{\pi}\int_0^t s^{-3/2}\left(\frac{s}{s+1}\right)ds && (295)\\
&= \frac{1}{\pi}\int_0^t \frac{\sqrt{s}}{s(s+1)}ds && (296)\\
&= \frac{2\arctan\bigl(\sqrt{t}\bigr)}{\pi} && (297)
\end{aligned}$$

It follows that

$$\mathbb{P}_0(R = 1+t) = \frac{\partial}{\partial t}\left(\mathbb{P}_0(R \le 1+t)\right) \tag{298}$$
$$= \frac{\partial}{\partial t}\left(\frac{2\arctan\left(\sqrt{t}\right)}{\pi}\right) \tag{299}$$
$$= \frac{2}{\pi}\cdot\frac{\partial}{\partial t}\left(\arctan\left(\sqrt{t}\right)\right) \tag{300}$$
$$= \frac{2}{\pi}\cdot\frac{1}{2t^{1/2}(t+1)} \tag{301}$$
$$= \frac{1}{\pi t^{1/2}(t+1)} \tag{302}$$

as desired. □

Problem 7.2.3.

Proof. First, apply Markov property on Theorem 7.2.5, we get $\inf\{t \in (a,b) : B_t = B_a\} = a$ almost surely. It follows there exists a decresing sequence $\{t_n\}$ with $t_n \downarrow a$ and $B_{t_n} = B_a$.

Similarly, apply Markov property on Theorem 7.2.4, we get $\inf\{t \in (a,b) : B_t > B_a\} = a$ almost surely, so that there exists a decreasing sequence $\{s_n\}$ with $s_n \downarrow a$ and $B_{s_n} > B_a$.

With the right arrangement of subsequence, we can form a strictly decreasing sequence with

$$t_{n_1} > s_{n_1} > t_{n_2} > s_{n_2} > \cdots > a \tag{303}$$

and $B_{t_k} = B_a, B_{s_k} > B_a$ for $k \in \mathbb{N}$. By continuity of Brownian path, there must exists some local maxima within each $(t_{n_{k+1}}, t_{n_k})$ interval, say $M_{n_k} \in (t_{n_{k+1}}, t_{n_k})$. By construction, $M_{n_k} \downarrow a$, as desired. □

Problem 7.4.3.

Proof. (a) For the set up, we define the stopping time $S = \inf\{s < t : B_s = a\}$ and denote

$$Y_s(\omega) = \begin{cases} 1 & s < t \text{ and } u < \omega(t-s) < v \\ 0 & \text{o.w.} \end{cases} \tag{304}$$

and

$$Y'_s(\omega) = \begin{cases} 1 & s < t \text{ and } 2a - v < \omega(t-s) < 2a - u \\ 0 & \text{o.w.} \end{cases} \tag{305}$$

By strong Markov property, Theorem 7.3.9, on the event $\{S < \infty\}$, we have

$$\mathbb{E}_0\left(Y_S \circ \theta_S \mid \mathcal{F}_S\right) = \mathbb{E}_{B_S} Y_S = \mathbb{E}_a Y_S \tag{306}$$
$$\mathbb{E}_0\left(Y'_S \circ \theta_S \mid \mathcal{F}_S\right) = \mathbb{E}_{B_S} Y'_S = \mathbb{E}_a Y'_S \tag{307}$$

By symmetry of normal distribution, we have $\mathbb{E}_a Y_S = \mathbb{E}_a Y'_S$, so that the above two equations are the same. On event $\{S < \infty\}$, it immediately follows that

$$\mathbb{P}_0\left(T_a < t, u < B_t < v\right) = \mathbb{E}_0\left(Y_S \circ \theta_S\right) \tag{308}$$
$$= \mathbb{E}_0\left(\mathbb{E}_0\left(Y_S \circ \theta_S \mid \mathcal{F}_S\right)\right) \tag{309}$$
$$= \mathbb{E}_0\left(\mathbb{E}_0\left(Y'_S \circ \theta_S \mid \mathcal{F}_S\right)\right) \tag{310}$$
$$= \mathbb{E}_0\left(Y'_S \circ \theta_S\right) \tag{311}$$
$$= \mathbb{P}_0\left(2a - v < B_t < 2a - u\right) \tag{312}$$

as desired.

(b) Send $u, v \to x$, we get

$$\mathbb{P}_0\left(T_a < t, B_t = x\right) = \mathbb{P}_0\left(B_t = 2a - x\right) = p_t(0, 2a - x) \tag{313}$$

(c) Since $\{T_a < t\} = \{M_t > a\}$, then part (b) implies that

$$\mathbb{P}_0\left(M_t > a, B_t = x\right) = \mathbb{P}_0\left(B_t = 2a - x\right) = p_t(0, 2a - x) \tag{314}$$

It follows that

$$\mathbb{P}_0\left(M_t \le a, B_t = x\right) = 1 - p_t(0, 2a - x) \tag{315}$$

so that

$$f_{(M_t, B_t)}(a, x) = \frac{\partial}{\partial a}\left(1 - p_t(0, 2a - x)\right) \tag{316}$$

$$= \frac{\partial}{\partial a}\Big(1 - (2\pi t)^{-1/2}\exp\big(-(2a - x)^2/2t\big)\Big) \tag{317}$$

$$= \frac{\sqrt{\frac{2}{\pi}}(2a - x)e^{-\frac{(x-2a)^2}{2t}}}{t^{3/2}} \tag{318}$$

$$= \frac{2(2a - x)}{\sqrt{2\pi t^3}}e^{-(2a-x)^2/2t} \tag{319}$$

as desired.

□

Problem 7.4.4.

Proof. We compute it directly, consider:

$$\mathbb{P}_0(A_{s,t}) = 2\int_0^\infty p_s(0, x)\mathbb{P}_x(T_0 \le t - s)dx \tag{320}$$

$$= 2\int_0^\infty (2\pi s)^{-1/2}\, e^{-x^2/2s}\mathbb{P}_x\left(T_0 \le t - s\right)dx \tag{321}$$

$$= 2\int_0^\infty (2\pi s)^{-1/2}\, e^{-x^2/2s}\mathbb{P}_0\left(T_x \le t - s\right)dx \tag{322}$$

$$= 2\int_0^\infty (2\pi s)^{-1/2}\, e^{-x^2/2s}\left(\int_0^{t-s}(2\pi r^3)^{-1/2}xe^{-x^2/2r}dr\right)dx \tag{323}$$

$$= \frac{2}{\sqrt{2\pi s}\cdot\sqrt{2\pi}}\int_0^\infty e^{-x^2/2s}\left(\int_0^{t-s} r^{-3/2}xe^{-x^2/2r}dr\right)dx \tag{324}$$

$$= \frac{1}{\pi\sqrt{s}}\int_0^{t-s} r^{-3/2}\left(\int_0^\infty x\exp\big(-\left(1/2r + 1/2s\right)x^2\big)\,dx\right)dr \tag{325}$$

$$= \frac{1}{\pi\sqrt{s}}\int_0^{t-s} r^{-3/2}\left(\frac{sr}{s + r}\right)dr \tag{326}$$

$$= \frac{1}{\pi\sqrt{s}}\int_0^{t-s}\frac{s\sqrt{r}}{r(r + s)}dr \tag{327}$$

$$= \frac{1}{\pi\sqrt{s}}\cdot\left(\frac{2s\sqrt{t/s - 1}}{\sqrt{t - s}}\cdot\arctan\left(\sqrt{t/s - 1}\right)\right) \tag{328}$$

$$= \left(\frac{1}{\pi\sqrt{s}}\cdot\frac{\frac{2s}{\sqrt{s}}\cdot\sqrt{t - s}}{\sqrt{t - s}}\right)\cdot\arctan\left(\sqrt{t/s - 1}\right) \tag{329}$$

$$= \frac{2}{\pi}\cdot\arctan\left(\sqrt{\frac{t - s}{s}}\right) \tag{330}$$

Finally, recall the well-known arctan identity

$$\arctan(x) = \arccos\left(\frac{1}{\sqrt{1+x^2}}\right) \tag{331}$$

that holds for all $x \geq 0$. Since $\sqrt{\frac{t-s}{s}} \geq 0$, we get

$$\frac{2}{\pi} \cdot \arctan\left(\sqrt{\frac{t-s}{s}}\right) = \frac{2}{\pi} \cdot \arccos\left(\frac{1}{\sqrt{1+\frac{t-s}{s}}}\right) \tag{332}$$

$$= \frac{2}{\pi} \cdot \arccos\left(\frac{1}{\sqrt{\frac{t}{s}}}\right) \tag{333}$$

$$= \frac{2}{\pi} \cdot \arccos\left(\sqrt{\frac{s}{t}}\right) \tag{334}$$

as desired. □

Remark 1. *The hint provided by Durrett is incorrect, note that* s *is missing on the* exp *function:*

$$\mathbb{P}_0(A_{s,t}) = 2\int_0^\infty (2\pi s)^{-1/2} e^{-x^2/2s} \mathbb{P}_x (T_0 \leq t-s)\, dx \tag{335}$$