Math 6490 Midterm I review

Niuniu Zhang

September 10, 2023

1 Conditional Expectation

Definition 1 (conditional expectation). $\mathbb{E}(X \mid \mathcal{F}) := Y \iff Y \in \mathcal{F}$ *and* $\forall A \in \mathcal{F}, \int_A X dP = \int_A Y dP$

Theorem 1 (4.1.9). *(a)* $\mathbb{E}(aX + Y \mid \mathcal{F}) = a\mathbb{E}(X \mid \mathcal{F}) + \mathbb{E}(Y \mid \mathcal{F})$

(b) $X \leq Y \implies \mathbb{E}(X \mid \mathcal{F}) \leq \mathbb{E}(Y \mid \mathcal{F})$

(c) $X_n \geq 0, X_n \uparrow X, \mathbb{E}X < \infty \implies \mathbb{E}(X_n \mid \mathcal{F}) \uparrow \mathbb{E}(X \mid \mathcal{F})$

Theorem 2 (4.1.12). $\mathcal{F} \subset \mathcal{G}, \mathbb{E}(X \mid \mathcal{G}) \in \mathcal{F} \implies \mathbb{E}(X \mid \mathcal{F}) = \mathbb{E}(X \mid \mathcal{G})$

Theorem 3 (4.1.13 Tower property). $\mathcal{F}_1 \subset \mathcal{F}_2 \implies \mathbb{E}(\mathbb{E}(X \mid \mathcal{F}_2)\mathcal{F}_1) = \mathbb{E}(\mathbb{E}(X \mid \mathcal{F}_1)\mathcal{F}_2) = \mathbb{E}(X \mid \mathcal{F}_1)$

Theorem 4 (4.1.14). $X \in \mathcal{F}, \mathbb{E}|Y|, \mathbb{E}|XY| < \infty \implies \mathbb{E}(XY \mid \mathcal{F}) = X\mathbb{E}(Y \mid \mathcal{F})$

Theorem 5 ("minimizer" 4.1.15). $\mathbb{E}X^2 < \infty \implies Y := \mathbb{E}(X \mid \mathcal{F}) \in \mathcal{F} \implies \min\{E(X-Y)^2\}$

2 Martingale

Definition 2 (Martingale). *$\mathcal{F}_n$ is filtration, X_n is said to be adapted to $\mathcal{F}_n$ if $X_n \in \mathcal{F}_n$ for all n. X_n is martingale if*

(i) $\mathbb{E}|X_n| < \infty$

(ii) X_n *adapted to* $\mathcal{F}_n$

(iii) $\mathbb{E}(X_{n+1} \mid \mathcal{F}_n) = X_n$ *for all* n

Theorem 6 (4.2.4/4.2.5). *The following claim applies for super/sub-martingale and martingale: If X_n is a martingale, then for $n > m, \mathbb{E}(X_n \mid \mathcal{F}_m) = X_m$.*

Definition 3 (predictable sequence). *$H_n, n \geq 1$ if $H_n \in \mathcal{F}_{n-1}$. If you bet according to a gambling system, then winning at time n would be*

$$(H \cdot X)_n = \sum_{m=1}^{n} H_m (X_m - X_{m-1}) \tag{1}$$

N stopping time takes value in $\mathbb{N}$, then $\{N = n\} \in \mathcal{F}_n$ and $H_n = \mathbf{1}_{N \geq n}$ is predicable and

$$(H \cdot X)_n = \sum_{k=1}^{n} \mathbf{1}_{N \geq k} (X_k - X_{k-1}) \tag{2}$$

Theorem 7 (4.2.8 Predicable). *X_n super-martingale. If $H_n \geq 0$ is predictable and H_n bounded, then $(H \cdot X)_n$ is a super-martingale.*

Theorem 8 (4.1.10 Jensen). *If φ is convex and $\mathbb{E}|X|, \mathbb{E}|\varphi(X)| < \infty$, then*

$$\varphi\left(\mathbb{E}(X \mid \mathcal{F})\right) \leq \mathbb{E}\left(\varphi(X) \mid \mathcal{F}\right) \tag{3}$$

For example, $|\cdot|$ is convex.

Theorem 9 (4.2.11. Martingale convergence theorem). $\sup \mathbb{E}X_n^+ < \infty \implies \lim_{n\to\infty} X_n \to X$ *a.s.*, $\mathbb{E}|X| < \infty$

Theorem 10 (4.2.12/super-martingale). $X_n \geq 0 \implies \lim_{n\to\infty} X_n = X$ *a.s.*, $\mathbb{E}X \leq \mathbb{E}X_0$

Theorem 11 (4.3.1 Bounded increment). *X_n martingale with $|X_{n+1} - X_n| \leq M < \infty$, then either $\lim X_n$ exists and is finite or oscillate between $+\infty$ and $-\infty$.*

$$C = \{\lim X_n \text{ exists and is finite}\} \tag{4}$$
$$D = \{\limsup X_n = \infty \text{ and } \liminf X_n = -\infty\} \tag{5}$$

then $\mathbb{P}(C \cup D) = 1$.

Definition 4 (Galton-Watson process). *ξ_i^n IID, define a sequence Z_n, the number of individuals in the nth generation, $Z_0 = 1$, then*

$$Z_{n+1} = \begin{cases} \xi_1^{n+1} + \cdots + \xi_{Z_n}^{n+1} & \text{if } Z_n > 0 \\ 0 & \text{if } Z_n = 0 \end{cases} \tag{6}$$

Lemma 1 (4.3.9 Branching process). *$\mathcal{F}_n = \sigma(\xi), \mu = \mathbb{E}\xi, Z_n/\mu^n$ is non-negative martingale.*

add why $\mathbb{E}Z_n = \mu^n$

Theorem 12 (4.3.10 Sub-critical). $\mu < 1 \implies Z_n = 0$ *for all n large, so $Z_n/\mu^n \to 0$.*

Theorem 13 (4.3.11 Critical). $\mu = 1, p_1 = \mathbb{P}(\xi_i^m = 1) < 1 \implies Z_n = 0$ *for all n large.*

Definition 5 (Generating function). $\forall s \in [0,1], \varphi(s) = \sum_{k\geq 0} p_k s^k = \sum_{k\geq 0} \mathbb{P}(\xi_i^m = k)\, s^k$.

Theorem 14 (4.3.12 Supercritical). $\mu > 1, Z_0 = 1 \implies \mathbb{P}(Z_n = 0 \text{ for some } n) = \rho$, *the only solution of $\varphi(\rho) = \rho$ in $[0,1)$.*

Theorem 15 (4.3.13). *$W = \lim Z_n/\mu^n \not\equiv 0$ iff $\sum p_k k \log k < \infty$. $\sum k^2 p_k < \infty$ is sufficient for a nontrivial limit.*

Theorem 16 (4.4.2 Doob's inequality). *Let X_m be sub-martingale, then*

$$\bar{X}_n = \max_{0\leq m\leq n} X_m^+ \tag{7}$$

$\lambda > 0$ and $A = \{\bar{X}_n \geq \lambda\}$. Then

$$\lambda \mathbb{P}(A) \leq \mathbb{E}X_n \mathbf{1}_A \leq \mathbb{E}X_n^+ \tag{8}$$

Theorem 17 (4.4.4 L^p maximum inequality). *If X_n is a sub-martingale, then for $1 < p < \infty$*

$$\mathbb{E}(\bar{X}_n^p) \leq \left(\frac{p}{p-1}\right)^p \mathbb{E}(X_n^+)^p \tag{9}$$

Theorem 18 (4.4.6 L^p convergence theorem). $\sup \mathbb{E}|X_n|^p < \infty$ *with $p > 1$, then $X_n \to X$ a.s. and in L^p.*

Theorem 19 (4.4.7 Ortho of martingale increment). $\mathbb{E}X_n^2 < \infty, m \leq n, Y \in \mathcal{F}_m$ *with $\mathbb{E}Y^2 < \infty$, then $\mathbb{E}((X_n - X_m)Y) = 0$. If $\ell < m < n$, then $\mathbb{E}((X_n - X_m)(X_m - X_\ell)) = 0$.*

Definition 6 (Uniform integrability). *UI iff*

$$\lim_{M\to\infty}\left(\sup_{i\in I}\mathbb{E}\left(|X_i|;|X_i|>M\right)\right)=0 \tag{10}$$

Theorem 20 (U.I. equivalence 4.6.7). *U.I* $\iff$ *Converges a.s. and in* L^1 $\iff$ *converges in* L^1 $\iff$ *there exists an integrable r.v.* X *with* $X_n=\mathbb{E}\left(X\mid\mathcal{F}_n\right)$

Theorem 21 (4.6.8). $F_n\uparrow F_\infty$, *then* $\mathbb{E}(X\mid\mathcal{F}_n)\to\mathbb{E}(X\mid\mathcal{F}_\infty)$ *a.s. and in* L^1.

Theorem 22 (4.6.9 Levy's 0-1 law). $\mathcal{F}_n\uparrow\mathcal{F}_\infty, A\in\mathcal{F}_\infty$, *then* $\mathbb{E}\left(\mathbf{1}_A\mid\mathcal{F}_n\right)\to\mathbf{1}_A$ *a.s.*

Theorem 23 (4.8.1). X_n *U.I. implies* $X_{N\wedge n}$ *U.I.*

Theorem 24 (4.8.2). *(Check exercise)* $\mathbb{E}|X_N|<\infty$ *and* $X_n\mathbf{1}_{N>n}$ *U.I., then* $X_{N\wedge n}$ *U.I. and* $\mathbb{E}X_0\le\mathbb{E}X_N$.

Theorem 25 (OST). *Suppose* $X_{N\wedge n}$ *is a U.I martingale. Let* $X_\infty=\lim_{n\to\infty}X_{N\wedge n}$ *on the event* $\{N=\infty\}$, *Then* $\mathbb{E}[X_N]=\mathbb{E}[X_0]$.

Theorem 26 (4.8.3). X_n *U.I. implies for* $N\le\infty$, $\mathbb{E}X_0\le\mathbb{E}X_N\le\mathbb{E}X_\infty=\mathbb{E}\lim X_n$.

Theorem 27 (4.8.7 SSRW). $\mathbb{P}(\xi=1)=\mathbb{P}(\xi=-1)=1/2, S_0=x$ *and* $N=\min\left(n:S_n\notin(a,b)\right)$, *then*

$$\mathbb{P}_x(S_N=a)=\frac{b-x}{b-a}\quad\mathbb{P}_x(S_N=b)=\frac{x-a}{b-a}\quad\mathbb{E}_xN=(b-x)(x-a) \tag{11}$$

Theorem 28 (4.8.9 ASRW). *(practice)* $\mathbb{P}(\xi=1)=p,\mathbb{P}(\xi=-1)=q,S_0=x$ *and* $N=\min\left(n:S_n\notin(a,b)\right)$, *then*

(a) $\varphi(y)=(q/p)^t$, *then* $\varphi(S_n)$ *is martingale.*

(b) $T_z=\inf\{n:S_n=z\}$, *then for* $a<x<b$, $P_x(T_a<T_b)=\frac{\varphi(b)-\varphi(x)}{\varphi(b)-\varphi(a)}$ *and* $P_x(T_b<T_a)=\frac{\varphi(x)-\varphi(a)}{\varphi(b)-\varphi(a)}$

If $p>1/2$, *we get*

(c) $a<0,\mathbb{P}\left(\min_n S_n\le a\right)=\mathbb{P}\left(T_a<\infty\right)=\left(\frac{q}{p}\right)^{-a}$

(c) $b>0$, *then* $\mathbb{P}(T_b<\infty)=1$ *and* $\mathbb{E}T_b=\frac{b}{2p-1}$

3 Brownian motion

Definition 7 (BM). *(a) Independent increment, (b)* $B(s+t)-B(s)\sim\mathcal{N}(0,t)$, *(c) continuous*

Definition 8 (BM translation invariance). $\{B_t-B_0\}$ *independent and has the same law as BW with* $B_0=0$.

Definition 9 (BM scaling relation). $B_0=0$, *then* $B_{st}\overset{d}{=}t^{1/2}B_s$

Definition 10 (Markov property/non-rigorous). *If* $s\ge0$, *then* $B(t+s)-B(s),t\ge0$ *is a Brownian motion that is independent of what happened before time* s. *What happened before* s:

$$\mathcal{F}_s^o=\sigma\left(B_r:r\le s\right) \tag{12}$$

Infinitesimal peek at the future:

$$\mathcal{F}_s^+=\cap_{t>s}\mathcal{F}_t^o \tag{13}$$

$A\in\mathcal{F}_s^+$ *if* $A\in\mathcal{F}_{s+\epsilon}^+$ *for any* $\epsilon>0$.

Theorem 29 (7.2.3 Bluementhal's 0-1 law). $A\in\mathcal{F}_0^+$, *then for all* $x\in\mathbb{R}^d,\mathbb{P}_x(A)\in\{0,1\}$.

Theorem 30 (7.2.4). $\tau=\inf\{t\ge0:B_t>0\}$, *then* $\mathbb{P}_0\left(\tau=0\right)=1$.

Theorem 31 (7.2.5)**.** *$T_0 = \inf(t > 0 : B_t = 0)$ then $P_0(T_0 = 0) = 1$.*

Theorem 32 (7.2.6 Inversion symmetry)**.** *B_t starts at zero implies $X_t = tB(1/t)$ BM starts at zero.*

Theorem 33 (7.2.8)**.** *B_t start at zero then with probability one, we have*

$$\limsup_{t\to\infty} B_t/\sqrt{t} = \infty \quad \liminf_{t\to\infty} B_t/\sqrt{t} = -\infty \tag{14}$$

Lemma 2 (from class)**.** *$b > a > 0, T_a - T_b \perp\!\!\!\perp T_a$ and $T_b - T_a \overset{d}{=} T_{b-a}$*

Theorem 34 (from class/Reflection principle 7.4.2)**.** *$\mathbb{P}_0(T_a < t) = 2\mathbb{P}_0(B_t \geq a)$*

Theorem 35 (from class/Zero set of BM)**.** *$Z = \{t : B_t = 0\}$. $t \in Z$ is isolated means $\exists\epsilon > 0, (t-\epsilon, t+\epsilon) \cap Z = \{t\}$. $\mathbb{P}(Z$ has no isolated point$) = 1$.*