Math 6490 Final Review Sheet

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1 BM and SRW

Theorem 1. *Define* $M_t = \max_{s\in[0,t]} B_s$ *and* $X_t = M_t - B_t$*, then we have* $(X_t)_{t\geq 0} \stackrel{d}{=} (|B_t|)_{t\geq 0}$.

Theorem 2 (Durrett 7.5.3)**.** B_t *is a martingale w.r.t* $\mathcal{F}_t$*. If* $a < x < b$*, then* $\mathbb{P}_x(T_a < T_b) = (b-x)/(b-a)$.

Theorem 3 (Durrett 7.5.5)**.** *Let* $T = \inf\{t : B_t \notin (a,b)\}$*, where* $a < 0 < b$*, then* $\mathbb{E}_0 T = -ab$.

Theorem 4 (Skorokhod's representation theorem)**.** *If* $\mathbb{E}X = 0, \mathbb{E}X^2 < \infty$*, then there eixsts* T *for BM so that* $B_T \stackrel{d}{=} X$ *and* $\mathbb{E}T = EX^2$.

Remark 1 (How to find the coupling?)**.** *Consider the following example: Let* ξ *an random variable with*

$$\mathbb{P}[\xi = 1] = \mathbb{P}[\xi = -1] = \frac{1}{6} \quad \mathbb{P}[\xi = 2] = \mathbb{P}[\xi = -2] = \frac{1}{3} \tag{1}$$

Imagine we have a symmetric pair of levels, $(1,2)$ *and* $(-2,-1)$*. In essence, we are cooking up stopping times so that the exit probability align with the distribution of* ξ*. Our main tool is Theorem 2. By symmetry of* $(1,2)$ *and* $(-2,-1)$*, it suffices to only consider the former.*

For any $x \in (1,2)$*, we wish to let BM exit* 1 *with probability* $1/6$ *and* 2 *with probability* $1/3$*, that is*

$$\frac{1}{2} \cdot \mathbb{P}_x(T_1 < T_2) = \frac{1}{2} \cdot \frac{2-x}{2-1} = \frac{1}{6} \implies x = \frac{5}{3} \tag{2}$$

(half since we only get half the "picture") Thus, we know by picking BM starts at $5/4$ *or* $-5/4$ *will have the desired results. We denote the starting point as*

$$T_0 = \inf\{t : |B_t| = \frac{5}{3}\} \tag{3}$$

Suppose BM has hit T_0*, to move to* 2*, we define*

$$T_2 = \inf\{t \geq T_0 : B_t = \frac{6}{5} B_{T_0}\} \tag{4}$$

To move to 1*, we define*

$$T_1 = \inf\{t \geq T_0 : B_t = \frac{3}{5} B_{T_0}\} \tag{5}$$

Thus, we have obtained the desired stopping time

$$T := T_1 \wedge T_2 \tag{6}$$

Theorem 5 (Durrett 8.1.2)**.** X_n *IID with distribution* F*, mean zero and variance one. Let* $S_n = \sum_n X_i$*, then there exists* T_n *such that* $S_n \stackrel{d}{=} B_{T_n}$ *and* $T_n - T_{n-1}$ *are independent and identically distributed.*

Theorem 6 (Donsker's theorem/Skorohod coupling)**.** $S(n\cdot)/\sqrt{n} \Rightarrow B(\cdot)$.

Theorem 7. *Suppose* L *is continuous for all* $f \in C[0,\infty)$*, then* $L\left(S(n\cdot)/\sqrt{n}\right) \Rightarrow L\left(B(\cdot)\right)$.

Theorem 8 (SRW Reflection Principle, Durrett 4.9.1)**.** *If* $x, y > 0$*, then the number of path from* $(0,x)$ *to* (n,y) *that are zero at some time is equal to the number of path from* $(0,-x)$ *to* (n,y).

2 Stochastic Calculus

2.1 Ito's Fundamentals

Definition 1 ($\mathcal{H}^2$). *$\mathcal{H}^2 = \mathcal{H}^2[0,T] = L^2(dP \times dt)$ and $f \in \mathcal{H}^2$ iff $\mathbb{E}\left[\int_0^T f^2(\omega,t)dt\right] < \infty$.*

Definition 2 ($\mathcal{H}_0^2$). *$\mathcal{H}_0^2 \subseteq \mathcal{H}^2$ and are consisted of function of the form*

$$f(\omega,t) = \sum_{i=0}^{n-1} a_i(\omega)\mathbf{1}_{(t_i < t \le t_{i+1})} \tag{7}$$

Let $I : \mathcal{H}_0^2 \to L^2(dP)$ to be a continuous mapping, then the above becomes

$$I(f)(\omega) = \sum_{i=0}^{n-1} a_i(\omega)\{B_{t_i+1} - B_{t_i}\} \tag{8}$$

Definition 3 (Ito Integral).

$$X_t = \int_0^t B_s dB_s = \frac{1}{2}(B_t^2 - t) \tag{9}$$

Lemma 1 (Density of $\mathcal{H}_0^2$ in $\mathcal{H}^2$). *$f \in \mathcal{H}^2[0,T]$ iff $\exists f_n \in \mathcal{H}_0^2[0,T]$ s.t $f_n \xrightarrow{L^2} f$ and*

$$L^2(dP \times dt) := L^2[\Omega \times [0,T]] = \{f(\omega,t) \mid \mathbb{E}\int_0^T f^2(\omega,t)dt < \infty\} \tag{10}$$

Definition 4 ($\mathcal{L}^2_{\mathrm{LOC}}$). *The class $\mathcal{L}^2_{LOC} = \mathcal{L}^2_{LOC}[0,T]$ consists of the type of function $f : \Omega \times [0,T] \mapsto \mathbb{R}$ such that*

$$\mathbb{P}\left(\int_0^T f^2(\omega,t)dt < \infty\right) = 1 \tag{11}$$

Lemma 2 (Ito's Isometry, Steele 6.1). *For $f \in \mathcal{H}_0^2$, we have $\|I(f)\|_{L^2(dP)} = \|f\|_{L^2(dP\times dt)}$. Alternatively, we may write $\mathbb{E}I(f)^2 = \mathbb{E}\int_0^T f^2(\omega,s)ds$. For example, we get*

$$\mathbb{E}\left[\left(\int_0^t |B_s|^{1/2} dB_s\right)^2\right] = \mathbb{E}\left[\int_0^t |B_s| ds\right] \tag{12}$$

Definition 5 (Standard Process, Steele 8.1). *X_t is a standard process if it has the following representation*

$$X_t = x_0 + \int_0^t a(\omega,s)ds + \int_0^t b(\omega,s)dB_s \tag{13}$$

where

$$\mathbb{P}\left(\int_0^T |a|ds < \infty\right) = 1 \quad \mathbb{P}\left(\int_0^T b^2 ds < \infty\right) = 1 \iff \left(b \in L^2_{LOC}[0,T]\right) \tag{14}$$

Theorem 9 (Quadratic Variation of Standard Process, Steele 8.6). *Let X_t be a standard process with*

$$X_t = \int_0^t a(\omega,s)ds + \int_0^t b(\omega,s)dB_s \tag{15}$$

then its quadratic varation is

$$\langle X\rangle_t = \int_0^t b^2(\omega,s)ds \tag{16}$$

2.2 Useful propositions

Proposition 1 (Gaussian Integrals, Steele 7.6). *If $f \in C[0,T]$, then the process defined by $X_t = \int_0^t f(s)dB_s$ is a mean zero Gaussian process with indep' increments and covariance function $Cov(X_s, X_t) = \int_0^{s\wedge t} f^2(u)du$.*

Definition 6 (Local Martingale). *If a process M_t is adapted to $\mathcal{F}_t$, then M_t is called a local martingale provided there is a nondecreasing sequence $\{\tau_k\}$ such that $\tau_k \uparrow \infty$ with probability one and $M_{t\wedge\tau_k} - M_0$ is true martingale.*

Proposition 2 (L^2_{LOC} function to local martingale, Steele 7.7). *$f \in L^2_{LOC}$, then there exists a local martingale X_t such that*

$$\mathbb{P}\left(X_t(\omega) = \int_0^t f(\omega, s)dB_s\right) = 1 \tag{17}$$

with localizing sequnce to be

$$\tau_n(\omega) = \inf\{t : \int_0^t f^2(\omega, s)ds \geq n \text{ or } t \geq T\} \tag{18}$$

Proposition 3 (Exit Probability, Steele 7.8). *X_t, local martingale with $X_0 = 0$. Let $\tau = \inf\{t : X_t = A$ or $X_t = -B\}$ satisfies $\mathbb{P}(\tau < \infty) = 1$, then $\mathbb{E}(X_\tau) = 0$ and $\mathbb{P}(X_\tau = A) = \frac{B}{A+B}$.*

Proposition 4 (Doob's analog, Steele 7.9). *X_t local martingale and τ stopping time, then $Y_t = X_{t\wedge\tau}$ is also a local martingale.*

Proposition 5 (Loc to Hon, Steele 7.10). *X_t local martingale, and B is a constant such that $|X_t| \leq B$, then X_t martingale.*

Proposition 6 (Loc to Hon, Steele 7.11). *X_t non-negative local martingale with $\mathbb{E}|X_0| < \infty$ is also a super martingale. If $\mathbb{E}X_T = \mathbb{E}X_0$, then X_t is a martingale.*

Proposition 7 (Martingale PDE condition, Steele 8.1). *$f \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R})$ and*

$$\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2} f = 0 \tag{19}$$

then $X_t = f(t, B_t)$ is local martingale. If

$$\mathbb{E}\left[\int_0^T \{\frac{\partial f}{\partial x}\}^2(t, B_t)dt\right] < \infty \tag{20}$$

then X_t is martingale.

Proposition 8 (Martingale PDE condition for $\mathbb{R}^d$, Steele 8.3). *$f \in C^{1,2}(\mathbb{R}^+, \mathbb{R}^d)$ and $B_t \in \mathbb{R}^d$, then $f(t, B_t)$ is local martingale given*

$$f_t(t, x) + \frac{1}{2}\Delta f(t, x) = 0 \tag{21}$$

Consequently, for $f(B_t), B_t \in \mathbb{R}^d$, we have

$$\Delta f = 0 \tag{22}$$

iff $f(B_t)$ local martingale.

Corollary 1 (Quadratic Varation PDE condition, Class 04/10). *If $f_t + \frac{1}{2}f_{xx} = 0$, then $f(\langle Z\rangle_t, Z_t)$ is a local martingale.*

Theorem 10 (Martingale Representation Theorem, Steele 12.3)**.** *X_t is a martingale w.r.t $\mathcal{F}_t$. If there exists a T such that $\mathbb{E}(X_T^2) < \infty$, then there is a $\phi \in \mathcal{H}^2[0,T]$ such that*

$$X_t = \int_0^t \phi(\omega, s) dB_s \tag{23}$$

The above holds for local too.

Theorem 11 (Levy's Representation Theorem, Steele 12.4)**.** *$\phi \in L^2_{LOC}[0,T]$ and*

$$X_t = \int_0^t \phi(\omega, s) dB_s \tag{24}$$

If we have

$$\mathbb{P}\left(\int_0^\infty \phi^2 ds = \infty\right) = 1 \quad \tau_t := \inf\left(u : \int_0^u \phi^2 ds \geq t\right) \tag{25}$$

then X_{τ_t} is BM.

Theorem 12 (BMC, Steele 12.5)**.** *Suppose M_t is a martingale. If $\mathbb{E}M_t^2 < \infty$ and $\langle M \rangle_t = t$, then M_t is a standard BM.*

Remark 2 (What if $\langle Z \rangle_t \neq t$, class 04/10)**.** *Assume $\langle Z \rangle_t \nearrow \infty$. Define τ_t to be the first time $\langle Z \rangle_t = t$, then we have the following consequences*

(i) $\limsup Z_t = -\liminf Z_t = \infty$.

(ii) $Z_{\tau_t} = B_t$ *and* $\sup_{\{s:\langle Z \rangle_s \leq t\}} Z_s \stackrel{d}{=} \sup_{0 \leq s \leq t} B_s$.

(iii) $T_0 = 0, T_k = \inf\{t : |Z_t - Z_{T_k - 1}| = 1\}$, *then Z_{τ_k} is BM by HW.*

(iv) Define

$$\sigma_M = \inf\{t : Z_s = M\} \quad \sigma'_M = \inf\{t : B_s = M\} \tag{26}$$

then $\inf_{t \leq \sigma_M} Z_t \stackrel{d}{=} \inf_{t \leq \sigma'_M} B_t$.

Remark 3 (L operator, Class 04/12)**.** *(i) (**Space only**): Let*

$$dX_t = \sigma(X_t) dB_t + \mu(X_t) dt \tag{27}$$

then

$$(dX_t)^2 = \sigma(X_t)^2 (dB_t)^2 = \sigma(X_t)^2 dt \tag{28}$$

Apply Ito's formula, we have

$$\begin{aligned} df(X_t) &= f'(X_t) dX_t + \frac{1}{2} f''(X_t)(dX_t)^2 &(29)\\ &= f'(X_t)\{\sigma(X_t) dB_t + \mu(X_t) dt\} + \frac{\sigma(X_t)^2}{2} f''(X_t) dt &(30)\\ &= f'\sigma dB_t + \{f'\mu + \frac{\sigma^2}{2} f''\} dt &(31) \end{aligned}$$

Observe that $f'\sigma \in L^2_{LOC}$, by Prop 7.7, should we want $f(X_t)$ to be a local martingale, the other term must be gone. Recall L operator is defined to be

$$Lf = f'\mu + \frac{\sigma^2}{2} f'' \tag{32}$$

then we may conclude that $Lf = 0$ iff $f(X_t)$ is a local martingale.

(ii) ***(Space and time)****: Let*

$$dX_t = \sigma(t, X_t)dB_t + \mu(t, X_t)dt \tag{33}$$

then

$$(dX_t)^2 = \sigma(t, X_t)^2 dt \tag{34}$$

Similarly, by Ito's formula (space and time), we get

$$df(t, X_t) = f_t dt + f_x dX_t + \frac{1}{2} f_{xx} (dX_t)^2 \tag{35}$$

$$= f_t dt + f_x\{\sigma dB_t + \mu dt\} + \frac{1}{2} f_{xx}\sigma^2 dt \tag{36}$$

$$= \sigma f_x dB_t + \{f_t + \mu f_x + \frac{\sigma^2}{2} f_{xx}\}dt \tag{37}$$

Note that $\sigma f_x \in L^2_{LOC}$, then $f(t, X_t)$ is a martingale iff the other terms are gone. In particular, note that

$$f_t + \mu f_x + \frac{\sigma^2}{2} f_{xx} = 0 \iff (dt + L)f = 0 \tag{38}$$

Theorem 13 (Existence and Uniqueness, Steele 9.1)**.** *Let $dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dB_t$ with $X_0 = x_0$ satisfy*

$$|\mu(t, x) - \mu(t, y)|^2 + |\sigma(t, x) - \sigma(t, y)|^2 \leq K|x - y|^2 \tag{39}$$

$$|\mu(t, x)|^2 + |\sigma(t, x)|^2 \leq K\left(1 + |x|^2\right) \tag{40}$$

then there exists a solution X_t that is uniformly bounded in L^2. If X_t, Y_t are both continuous L^2 bounded solution, then they are the same almost surely.

Warning: sign of **drift** would change the form of M_t.

Theorem 14 (Simplest Girsanov Theorem, Steele 13.1)**.** *B_t is $\mathbb{P}$-BM and $\mathbb{Q}$ is induced by*

$$X_t = B_t + \mu t \tag{41}$$

then every bounded Borel measurable function W on $C[0, T]$ satisfies

$$\mathbb{E}_{\mathbb{Q}}(W) = \mathbb{E}_{\mathbb{P}}(W M_T) \tag{42}$$

where M_t is $\mathbb{P}$-martingale defined by

$$M_t = \exp\bigl(\mu B_t - \mu^2 t/2\bigr) \tag{43}$$

Theorem 15 (Removing Drift, Steele 13.2)**.** *$\mu(\omega, t)$ is a bounded, adapted process on $[0, T]$, B_t is a $\mathbb{P}$-BM, and X_t given by*

$$X_t = B_t + \int_0^t \mu(\omega, s)ds \tag{44}$$

The process M_t defined by

$$M_t = \exp\left(-\int_0^t \mu(\omega, s)dB_s - \frac{1}{2}\int_0^t \mu^2(\omega, s)ds\right) \tag{45}$$

is a $\mathbb{P}$-martingale and the product $X_t M_t$ is also a $\mathbb{P}$-martingale. Finally, if $\mathbb{Q}$ denotes the measure on $C[0, T]$ defined by

$$\mathbb{Q}(A) = \mathbb{E}_{\mathbb{Q}}[\mathbf{1}_A] = \mathbb{E}_{\mathbb{P}}[\mathbf{1}_A M_T] \tag{46}$$

then X_t is a $\mathbb{Q}$-Brownian motion on $[0, T]$.

2.3 Ito's formula

Theorem 16 (Simple Ito's formula, Steele 8.1). *$f \in C^2(\mathbb{R})$ and $f(B_t)$, then*

$$f(B_t) = f(0) + \int_0^t f'(B_s)dB_s + \frac{1}{2}\int_0^t f''(B_s)ds \tag{47}$$

$$df(B_t) = f'(B_s)dB_s + \frac{1}{2}f''(B_s)d_s \tag{48}$$

Note the general fact that $(dB_s)^2 = ds, (dB_s)^n = 0$ for $n > 2$.

Theorem 17 (Ito's formula with time and space, Steele 8.2). *$f \in C^{1.2}(\mathbb{R}^+ \times \mathbb{R})$ and $f(t, B_t)$, then we have*

$$f(t,B_t) = f(0) + \int_0^t \frac{\partial f}{\partial x}(s,B_s)dB_s + \int_0^t \frac{\partial f}{\partial t}(s,B_s)ds + \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x^2}(s,B_s)ds \tag{49}$$

$$df(t,B_t) = f_x(s,B_s)dB_s + f_t(s,B_s)ds + \frac{1}{2}f_{xx}(s,B_s)ds \tag{50}$$

Theorem 18 (Vector Ito's formula, Steele 8.3). *$f \in C^{1,2}(\mathbb{R}^+, \mathbb{R}^d)$ and $B_t \in \mathbb{R}^d$, then*

$$df(t,B_t) = f_t(t,B_t)dt + \nabla f(t,B_t)dB_t + \frac{1}{2}\Delta f(t,B_t)dt \tag{51}$$

Theorem 19 (Local Martingale, Class 04/05). *Let $Z_t = \int_0^t b(\omega,s)dB_s, b \in L^2_{LOC}$ and $f(Z_t)$. From the general fact, we have*

$$dZ_t = bdB_t \quad (dZ_t)^2 = (bdB_t)^2 = b^2dt \tag{52}$$

then we have

$$df(Z_t) = f'(Z_s)dZ_s + \frac{1}{2}f''(Z_s)(dZ_s)^2 \tag{53}$$

$$= f'(Z_s)bdB_s + \frac{1}{2}f''(Z_s)b^2ds \tag{54}$$

Alternatively, we have

$$f(Z_t) = f(Z_0) + \int_0^t f'(Z_s)bdB_s + \frac{1}{2}\int_0^t f''(Z_s)b^2ds \tag{55}$$

Theorem 20 (Standard Process, Steele 8.4). *$f \in C^{1,2}(\mathbb{R}^+, \mathbb{R})$ and*

$$X_t = \int_0^t a(\omega,s)ds + \int_0^t b(\omega,s)dB_s \tag{56}$$

then $dX_s = bdB_s, (dX_s)^2 = b^2ds$, so that

$$f(t,X_t) = f(0) + \int_0^t f_t(s,X_s)ds + \int_0^t f_x(s,X_s)dX_s + \frac{1}{2}\int_0^t f_{xx}(s,X_s)(dX_t)^2 \tag{57}$$

$$= f(0) + \int_0^t f_t(s,X_s)ds + \int_0^t f_x(s,X_s)dX_s + \frac{1}{2}\int_0^t f_{xx}(s,X_s)b^2ds \tag{58}$$

$$df(t,X_t) = f_t(s,X_s)ds + f_x(s,X_s)dX_s + \frac{1}{2}f_{xx}(s,X_s)b^2ds \tag{59}$$

Theorem 21 (Quadratic Variation, Class 04/10)**.**

$$f\left(\langle Z\rangle_t, Z_t\right) = f(0, Z_0) + \int_0^t f_x(\langle Z\rangle_t, Z_t) dZ_t + \int_0^t f_t(\langle Z\rangle_t, Z_t) dt + \frac{1}{2}\int_0^t f_{xx}(\langle Z\rangle_t, Z_t)\left(dZ_t\right)^2 \tag{60}$$

$$= f(0, Z_0) + \int_0^t f_x(\langle Z\rangle_t, Z_t) b dB_t + \int_0^t f_t(\langle Z\rangle_t, Z_t) dt + \frac{1}{2}\int_0^t f_{xx}(\langle Z\rangle_t, Z_t) b^2 dt \tag{61}$$

$$= f(0, Z_0) + \int_0^t f_x(\langle Z\rangle_t, Z_t) b dB_t + \int_0^t f_t(\langle Z\rangle_t, Z_t) dt + \frac{1}{2}\int_0^t f_{xx}(\langle Z\rangle_t, Z_t) b^2 dt \tag{62}$$

$$df(\langle Z\rangle_t, Z_t) = f_x(\langle Z\rangle_t, Z_t) b dB_t + f_t(\langle Z\rangle_t, Z_t) dt + \frac{1}{2} f_{xx}(\langle Z\rangle_t, Z_t) b^2 dt \tag{63}$$